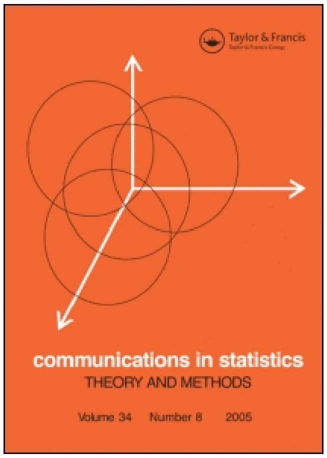

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Colin M. Ramsay [a]
[a] Department of Finance, University of Nebraska-Lincoln, Lincoln, Nebraska, USA

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Distributions and Applications

The Distribution of Sums of Certain I.I.D. Pareto Variates

COLIN M. RAMSAY

Department of Finance, University of Nebraska–Lincoln, Lincoln, Nebraska, USA

Though the Pareto distribution is important to actuaries and economists, an exact expression for the distribution of the sum of n i.i.d. Pareto variates has been difficult to obtain in general. This article considers Pareto random variables with common probability density function (pdf) $f(x) = (\alpha/\beta)(1 + x/\beta)^{\alpha+1}$ *for* $x > 0$, *where* $\alpha = 1, 2, \ldots$ *and* $\beta > 0$ *is a scale parameter. To date, explicit expressions are known only for a few special cases:* (i) $\alpha = 1$ *and* $n = 1, 2, 3$; (ii) $0 < \alpha < 1$ *and* $n = 1, 2, \ldots$; *and* (iii) $1 < \alpha < 2$ *and* $n = 1, 2, \ldots$. *New expressions are provided for the more general case where* $\beta > 0$, *and* α *and* n *are positive integers. Laplace transforms and generalized exponential integrals are used to derive these expressions, which involve integrals of real valued functions on the positive real line. An important attribute of these expressions is that the integrands involved are non oscillating.*

Keywords Contour integration; Convolution; Exponential integral; Laplace transform; Pareto distribution.

Mathematics Subject Classification Primary 44A10, 65R10; Secondary 44A35.

1. Introduction

1.1. *A Problem from Actuarial Risk Theory*

The Pareto distribution is an important statistical distribution to economists and actuaries. In economics the Pareto distribution is used traditionally to model the income distribution of populations; see, for example, Arnold (1983, Ch. 1 and 2), Johnson and Kotz (1970, Ch. 19), Lambert (1993), and Mandelbrot (1960, 1963). Actuaries use the Pareto distribution to model catastrophic losses in an insurance portfolio; see, for example, Hogg and Klugman (1984) or Daykin et al. (1994, Ch. 3.3.7). The convolution of Pareto distributions may then be needed to determine the tail behavior of the distribution of aggregate claims, or the probability of ruin of the

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Address correspondence to Colin M. Ramsay, Department of Finance, University of Nebraska–Lincoln, Lincoln, NE 68588-0490, USA; E-mail: cramsay@unlnotes.unl.edu

portfolio. For example, the risk surplus process of the classical compound Poisson risk model of actuarial risk theory, $U(t)$, is defined as:

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i$$

where $u \geq 0$ is the initial risk surplus at $t = 0$, $N(t)$ is a time homogenous Poisson process with intensity λ, and the X_i's are claim sizes that are independent and identically distributed non negative random variables with cumulative distribution function (cdf) $F_X(x)$, finite mean $\mu_X = \mathbb{E}[X_i]$, and $c = (1+\theta)\lambda\mu_X$ is the premium rate with loading $\theta \geq 0$. The probability of ultimate non ruin given an initial surplus of $u \geq 0$ is $\Phi(u)$ where

$$\Phi(u) = \Pr[U(t) \geq 0 \text{ for all } t > 0 \,|\, U(0) = u].$$

It is well known that $\Phi(u)$ satisfies the following equation:

$$\Phi(u) = \sum_{k=0}^{\infty} \left(\frac{\theta}{1+\theta}\right)\left(\frac{\theta}{1+\theta}\right)^k F_e^{*k}(u) \tag{1}$$

where $F_e^{*k}(y)$ is the kth convolution of $F_e(y)$ and

$$F_e(y) = \frac{1}{\mu_X} \int_0^y (1 - F_X(x))dx,$$

which is sometimes called the equilibrium cdf. Now, if the claims are Pareto, i.e., $F_X(x)$ is a Pareto cdf, then $F_e(x)$ is also a Pareto cdf. Thus Eq. (1) requires the computation of the convolution of Pareto variates. See, for example, Bowers et al. (1997, Ch. 13.6) or Klugman et al. (2004, Ch. 8.4) for more details of this approach to solving the ruin problem. An alternative approach to deriving $\Phi(u)$ when $F_X(x)$ is a Pareto cdf is given by Ramsay (2003).

1.2. *Some Known Results*

There are two equivalent ways of defining a Pareto pdf:

$$f(x) = \frac{\alpha}{\beta}\left(\frac{\beta}{x+\beta}\right)^{\alpha+1} \quad \text{for } x > 0 \text{ and } \alpha, \beta > 0; \tag{2}$$

$$g(x) = \frac{\alpha}{\beta}\left(\frac{\beta}{x}\right)^{\alpha+1} \quad \text{for } x > \beta \text{ and } \alpha, \beta > 0. \tag{3}$$

In each case β is a scale parameter. Clearly, if X and Y have pdfs given by $f(x)$ and $g(x)$, respectively, then $Y = X + \beta$. Similarly, if $\{X_i\}$ and $\{Y_i\}$ are sequences of independent and identically distributed (i.i.d.) Pareto random variables with pdf $f(x)$ and $g(x)$, respectively, then, for fixed n,

$$\Pr\left[\sum_{i=1}^{n} Y_i \leq x\right] = \Pr\left[\sum_{i=1}^{n} X_i \leq x - n\beta\right].$$

Hence one can choose either $f(x)$ or $g(x)$ to study the distribution of the sum of i.i.d. Pareto variates.

Though the Pareto distribution itself is mathematically simple, it is difficult to determine the distribution of the sum of two or more i.i.d. Pareto random variables. What is generally known is that the distribution of sums of Pareto random variables behaves like a Pareto distribution in the tail. Specifically, Feller (1971, Ch. 8, pp. 268–272) has shown that as $x \to \infty$,

$$\Pr\left[\sum_{i=1}^{n} X_i > x\right] \sim n\left(1+\frac{x}{\beta}\right)^{-\alpha} L(x)$$

where $L(x)$ is a slowly varying function at infinity.[1] Roehner and Winiwarter (1985) gave expressions for the asymptotic behavior of a finite sum of non i.i.d. Pareto random variables, and the limiting density of the "renormalized" sum of n i.i.d. Pareto random variables as $n \to \infty$, i.e., the asymptotic density of

$$\frac{X_1 + X_2 + \cdots + X_n}{a_n} - b_n \sim \frac{1}{\alpha\pi}\int_0^\infty e^{-au}\cos(u^{1/\alpha}s + bu)u^{(1/\alpha)-1}du$$

where $a = -x_0^\alpha \alpha\Gamma(-\alpha)\cos(\alpha\pi/2)$ and $b = x_0^\alpha \alpha\Gamma(-\alpha)\sin(\alpha\pi/2)$ provided the coefficients a_n and b_n are chosen so that $a_n = n^{1/\alpha}$ and $b_n = nE[X_1]$ if $1 < \alpha < 2$, and $a_n = n^{1/\alpha}$ and $b_n = 0$ if $0 < \alpha < 1$ in Eq. (3).

Explicit results are known about the distribution of sums of Pareto random variables in certain special cases. Hagstroem (1960) used $\alpha = \beta = 1$ in Eq. (3), i.e., $g(x) = 1/x^2$ and derived exact results for the case where $n = 2$ and $n = 3$. In particular, Hagstroem showed that if $s_n(x) = \Pr[\sum_{i=1}^{n} Y_i > x]$, with $x > n$, then

$$s_1(x) = \frac{1}{x} \quad \text{for } x > 1;$$
$$s_2(x) = \frac{2}{x} + \frac{2\log(x-1)}{x^2} \quad \text{for } x > 2;$$

and

$$\begin{aligned} s_3(x) = {} & \frac{3}{x} + \frac{6(x-2)\log(x-2)}{x^3(x-1)} + \frac{4}{x^3}\log(x-1)\log(x-2) \\ & + \frac{2}{x^3}\left[(\log(x-1))^2 - (\log 2)^2\right] \\ & - \frac{4}{x^3}\left\{L_{1,0}\left(\frac{1}{(x-1)}, \frac{1}{(x-1)}\right) + L_{1,0}\left(\frac{1}{(x-1)}, \frac{1}{2}\right)\right\} \quad \text{for } x > 3, \end{aligned}$$

where, for $0 \le a, b, \le 1$,

$$L_{q,r}(a,b) = \int_b^{1-a} \frac{-\log v}{(1-v)^q v^r} dv.$$

[1] $L(x)$ is said to be slowly varying at infinity if, for fixed $t > 0$, $L(tx)/L(x) \to 1$ as $x \to \infty$.

Brennan et al. (1968) and Blum (1970) obtain a series expansion of the probability density function of $\sum_{i=1}^{n} Y_i$ for the case of $\beta = 1$, $0 < \alpha < 2$ and $\alpha \neq 1$ in Eq. (3). They proved that the pdf of $\sum_{i=1}^{n} Y_i$ is $g_n(x)$ (i.e., the n-fold convolution of $g(x)$), which is given by

$$g_n(x) = \frac{-1}{\pi} \sum_{j=1}^{n} \binom{n}{j} (-\Gamma(1-\alpha))^j \sin(\pi\alpha j) \sum_{m=0}^{\infty} C_{n-j,m} \frac{\Gamma(m+\alpha j+1)}{x^{m+\alpha j+1}} \tag{4}$$

where $C_{k,m}$ is the mth coefficient in the series expansion of the kth power of the confluent hypergeometric function

$${}_1F_1(-\alpha, 1-\alpha, t) = \sum_{j=0}^{\infty} \binom{-\alpha}{j-\alpha} \frac{1}{j!} t^j,$$

i.e.,

$$\sum_{m=0}^{\infty} C_{k,m} t^m = ({}_1F_1(-\alpha, 1-\alpha, t))^k.$$

Blum cautions that computational difficulties may arise in attempts to use Eq. (4) to compute $g_n(x)$ for large values of n and certain ranges of x and α.

1.3. *Objectives*

As was pointed out above, the only known exact expressions for the pdf of the sum (convolution) of n i.i.d. Pareto random variables are those given by Brennan et al. (1968) and Blum (1970) given in Eq. (4). However, their results are valid only for a small range of values of α, namely, $0 < \alpha < 1$ and $1 < \alpha < 2$. To date, *no exact expression is known for the case where* (i) α is a positive integer ($\alpha = 1, 2, \ldots$) or (ii) $\alpha \geq 2$.

This article provides an exact expression for the pdf and the cdf in case (i) where α is a positive integer. The approach used is to invert the Laplace transform of the convolution equation using the complex inversion formula, i.e., the Bromwich integral. An attractive feature of this approach is the expressions for the pdf and cdf involve a single real integral along the positive real line and the integral is not of an oscillating kind.

Note that, in general, one could obtain the pdf and cdf of the nth convolution of i.i.d. random variables with common pdf $f(x)$ by *repeated* application of a numerical integration scheme to the well-known recursive convolution equation:

$$f_k(x) = \int_0^x f_{k-1}(x-u) f(u)\, du$$

and

$$F_k(x) = \int_0^x F_{k-1}(x-u) f(u)\, du$$

for $k = 2, 3, \ldots$, where f_k and $F_k(x)$ denotes the kth convolution, and $f_1(x) \equiv f(x)$ and $F_1(x) \equiv F(x)$. This recursive approach may yield good results but can be relatively slow, especially if n and x are large or many values of n and x are needed. This is because one must compute values of the intervening pdfs and cdfs for $n = 1, 2, \ldots, m-1$ and appropriate values of $y \leq x$, depending on the numerical integration scheme. The result provided in this article, however, yields the value of $f^{n*}(x)$ and $F^{n*}(x)$ directly without computing any values of the intervening pdfs or cdfs.

Because of the traditional notation for complex variables is $z = x + iy$, to avoid confusion, the function $f(t)$ is used to denote the pdf rather than $f(x)$.

2. Main Results

Consider the Pareto pdf in Eq. (2) with integral parameter $\alpha = m$, i.e.,

$$f(t) = \frac{m}{\beta}\left(1 + \frac{t}{\beta}\right)^{-(m+1)} \quad \text{for } t > 0 \text{ and } m = 1, 2, \ldots.$$

For complex z and $\mathrm{Re}(z) > 0$, define the Laplace transform of $f_n(t)$ as

$$f_n^*(z) = \int_0^\infty e^{-zt} f_n(t)dt = (f^*(z))^n.$$

It can easily be proved that $f^*(z)$ can be written as

$$f^*(z) = me^{\beta z}E_{m+1}(\beta z), \tag{5}$$

where $E_m(z)$ is the generalized exponential integral. For $m = 1, 2, \ldots$ and $\mathrm{Re}(z) > 0$, $E_m(z)$ is defined by

$$E_m(z) = \int_1^\infty \frac{e^{-zt}}{t^m}dt$$

and by analytic continuation elsewhere. (The exponential integral is related to the incomplete gamma function. See Abramowitz and Stegun, 1964, Ch. 5 for more on exponential integrals.) It follows that

$$f_n^*(z) = (me^{\beta z}E_{m+1}(\beta z))^n. \tag{6}$$

The inversion of $f_n^*(z)$ is obtained as the Bromwich integral

$$f_n(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{tz} f_n^*(z)dz, \quad t > 0$$

where $c > 0$ is an arbitrary constant large enough so that all of the singularities of $f_n^*(z)$ lie to the left of the vertical line $\mathrm{Re}(z) = c$. As $E_m(z)$ has a logarithmic branch cut along the negative real axis and a branch point at the origin, the Bromwich integral can be evaluated as a part of the integral in the counter-clockwise direction around the deformed contour Ω. Specifically, Ω is the positively oriented closed path consisting of (i) the vertical line $z(u) = c + iu$ where u goes from $-y$ to y, (ii) the large semi-circle C_R centered at the origin and with radius R lying to the left of

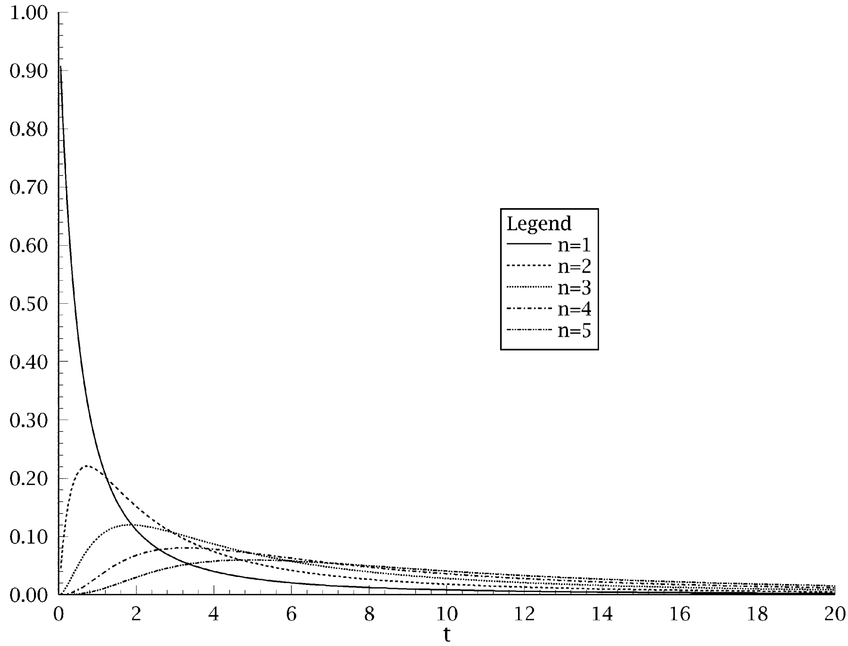

Figure 1. First 5 convolutions of the Pareto pdf $f(t) = (1 + t)^{-2}$.

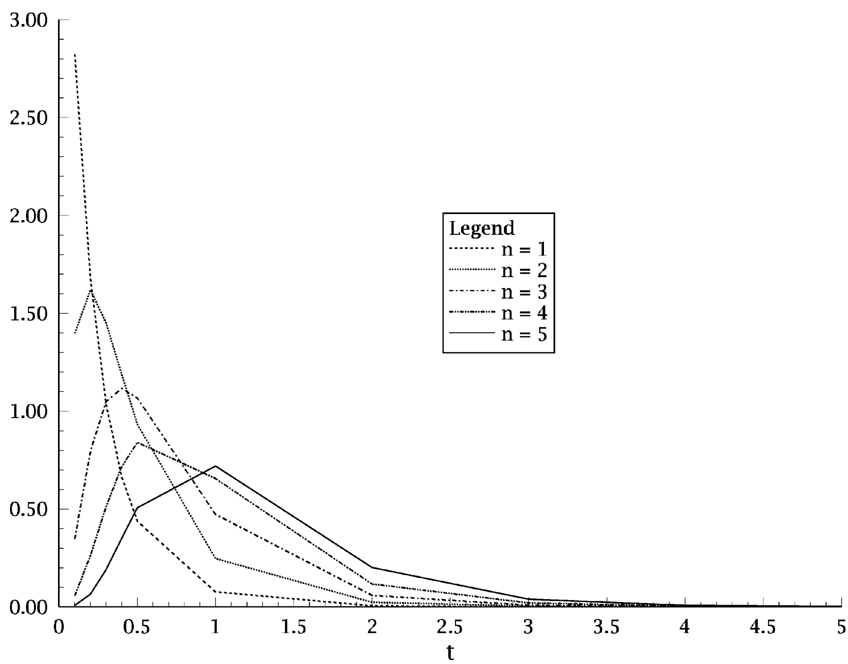

Figure 2. First 5 convolutions of the Pareto pdf $f(t) = 5(1 + t)^{-6}$.

the vertical line and passing through the points $c \pm iy$, (iii) the line $-R$ to $-r$ lying above the branch cut along the negative real axis, (iv) the small (almost) circle, C_r, about the origin with radius r, and (v) the line $-r$ to $-R$ lying below the branch cut.

However, as $me^{\beta z}E_{m+1}(\beta z)$ is analytic in Ω, it follows immediately that

$$\frac{1}{2\pi i}\int_{\Omega} e^{tz}(me^{\beta z}E_{m+1}(\beta z))^n dz = 0$$

as $r \to 0$ and $R \to \infty$. Also, as $e^z E_m(z) = O(1/z)$ as $|z| \to \infty$, the contribution from the large circle C_R is zero as $R \to \infty$. Likewise, the contribution from the circle around the origin (C_r) is easily seen to be zero as $r \to 0$. Hence, as $R \to \infty$ and $r \to 0$,

$$f_n(t) = \frac{1}{2\pi i}\int_0^{\infty} e^{-tx}(f_n^*(xe^{-i\pi}) - f_n^*(xe^{i\pi}))dx. \tag{7}$$

The definitions provided by Abramowitz and Stegun (1964, Ch. 5, Eqs. (5.1.7) and (5.1.12)) are generalized so that, for $x > 0$ and $m = 1, 2, 3, \ldots,$

$$E_m(xe^{\pm i\pi}) = -\mathrm{Ei}_m(x) \mp i\pi\frac{x^{m-1}}{(m-1)!} \tag{8}$$

where

$$\mathrm{Ei}_m(x) = \frac{x^{m-1}}{(m-1)!}\left[\gamma + \ln x - \sum_{r=1}^{m-1}\frac{1}{r}\right] + \sum_{\substack{r=0\\ r\neq m-1}}^{\infty}\frac{x^r}{(r-m+1)r!} \tag{9}$$

and $\gamma = 0.5772156649\ldots$ is the Euler constant. Substituting Eq. (8) into Eq. (6), and Eq. (6) into Eq. (7), and using the change of variable $v = \beta x$ yields

$$f_n(t) = \frac{(-m)^n}{2\pi i\beta}\int_0^{\infty} e^{-(1+\frac{t}{n\beta})nv}\left[\left(\mathrm{Ei}_{m+1}(v) - \pi i\frac{(v)^m}{m!}\right)^n - \left(\mathrm{Ei}_{m+1}(v) + \pi i\frac{(v)^m}{m!}\right)^n\right]dv. \tag{10}$$

But for real a and b,

$$(a - ib)^n - (a + ib)^n = -2i\sum_{r=0}^{\lfloor (n-1)/2\rfloor}(-1)^r\binom{n}{2r+1}a^{n-2r-1}b^{2r+1} \tag{11}$$

where $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x, hence

$$f_n(t) = \frac{1}{n\beta}\int_0^{\infty} e^{-(1+\frac{t}{n\beta})v}\varphi_{m,n}(v/n)dv \tag{12}$$

where, for $v > 0$, $\varphi_{m,n}(v)$ is defined as:

$$\varphi_{m,n}(v) = (-1)^{n+1}m^n\sum_{r=0}^{\lfloor (n-1)/2\rfloor}(-\pi^2)^r\binom{n}{2r+1}(\mathrm{Ei}_{m+1}(v))^{n-2r-1}\left(\frac{v^m}{m!}\right)^{2r+1}. \tag{13}$$

Table 1
Comparing sample values of $F_n(t)$ for $m = \beta = 1$ using Eq. (14) and simulation

	$n=2$			$n=3$			$n=4$			$n=5$		
t	Eq. (14)	SIMUL	ERROR	Eq. (14)	SIMUL	ERROR	Eq. (14)	SIMUL	ERROR	Eq. (14)	SIMUL	ERROR
0.1	0.00439	0.00424	3.60E − 02	0.00014	0.00014	4.10E − 02	0.00000	0.00000	1.60E − 01	0.00000	0.00000	1.40E + 01
0.2	0.01557	0.01534	1.50E − 02	0.00100	0.00094	6.60E − 02	0.00005	0.00004	1.40E − 01	0.00000	0.00000	4.80E − 01
0.3	0.03124	0.03097	8.80E − 03	0.00296	0.00287	3.10E − 02	0.00021	0.00021	6.60E − 03	0.00001	0.00001	2.00E − 01
0.4	0.04984	0.04952	6.30E − 03	0.00617	0.00613	6.00E − 03	0.00059	0.00057	3.20E − 02	0.00005	0.00004	1.90E − 01
0.5	0.07025	0.06989	5.20E − 03	0.01065	0.01070	4.30E − 03	0.00125	0.00122	2.90E − 02	0.00012	0.00012	3.60E − 02
1.0	0.17930	0.17924	3.20E − 04	0.04911	0.04913	5.20E − 04	0.01078	0.01083	4.50E − 03	0.00197	0.00203	3.00E − 02
2.0	0.36267	0.36331	1.70E − 03	0.16326	0.16393	4.10E − 03	0.06199	0.06237	6.20E − 03	0.02022	0.02018	1.60E − 03
3.0	0.48910	0.48978	1.40E − 03	0.27718	0.27806	3.20E − 03	0.13736	0.13807	5.20E − 03	0.06006	0.06021	2.60E − 03
4.0	0.57725	0.57786	1.00E − 03	0.37372	0.37423	1.30E − 03	0.21717	0.21780	2.90E − 03	0.11368	0.11399	2.80E − 03
5.0	0.64115	0.64174	9.20E − 04	0.45245	0.45321	1.70E − 03	0.29216	0.29302	3.00E − 03	0.17273	0.17305	1.80E − 03
10.0	0.80003	0.79979	3.00E − 04	0.67836	0.67869	4.80E − 04	0.55211	0.55298	1.60E − 03	0.43011	0.43116	2.40E − 03
20.0	0.89651	0.89648	3.10E − 05	0.83266	0.83297	3.70E − 04	0.76171	0.76251	1.10E − 03	0.68518	0.68563	6.60E − 04
30.0	0.93079	0.93085	6.10E − 05	0.88902	0.88911	1.10E − 04	0.84247	0.84262	1.80E − 04	0.79140	0.79162	2.90E − 04
40.0	0.94817	0.94857	4.20E − 04	0.91753	0.91793	4.30E − 04	0.88362	0.88424	7.00E − 04	0.84642	0.84702	7.10E − 04
50.0	0.95863	0.95901	3.90E − 04	0.93459	0.93499	4.30E − 04	0.90819	0.90879	6.70E − 04	0.87936	0.87971	4.00E − 04
75.0	0.97256	0.97283	2.70E − 04	0.95710	0.95748	4.00E − 04	0.94040	0.94088	5.10E − 04	0.92239	0.92283	4.70E − 04
100.0	0.97950	0.97973	2.30E − 04	0.96818	0.96848	3.20E − 04	0.95608	0.95648	4.10E − 04	0.94319	0.94351	3.40E − 04

Note: ERROR = |(Eq. (14) – SIMUL)/Eq. (14)|.

Table 2
Comparing sample values of $F_n(t)$ for $m = 5$ and $\beta = 1$ using Eq. (14) and simulation

	$n = 2$			$n = 3$			$n = 4$			$n = 5$		
t	Eq. (14)	SIMUL	ERROR	Eq. (14)	SIMUL	ERROR	Eq. (14)	SIMUL	ERROR	Eq. (14)	SIMUL	ERROR
0.1	0.08541	0.08509	3.80E − 03	0.01349	0.01359	7.30E − 03	0.00163	0.00161	1.20E − 02	0.00016	0.00016	2.30E − 02
0.2	0.24173	0.24191	7.20E − 04	0.07189	0.07220	4.30E − 03	0.01675	0.01674	6.30E − 04	0.00319	0.00331	3.50E − 02
0.3	0.39688	0.39787	2.50E − 03	0.16595	0.16644	2.90E − 03	0.05563	0.05597	6.10E − 03	0.01546	0.01544	1.30E − 03
0.4	0.52901	0.52973	1.40E − 03	0.27561	0.27654	3.40E − 03	0.11777	0.11848	6.10E − 03	0.04228	0.04228	9.30E − 05
0.5	0.63482	0.63542	9.40E − 04	0.38562	0.38629	1.70E − 03	0.19639	0.19685	2.30E − 03	0.08520	0.08551	3.60E − 03
1.0	0.89432	0.89403	3.20E − 04	0.76840	0.76840	7.30E − 06	0.60438	0.60520	1.40E − 03	0.43157	0.43278	2.80E − 03
2.0	0.98623	0.98639	1.70E − 04	0.96645	0.96671	2.70E − 04	0.93093	0.93138	4.90E − 04	0.87457	0.87504	5.50E − 04
3.0	0.99703	0.99711	8.40E − 05	0.99314	0.99337	2.30E − 04	0.98596	0.98608	1.20E − 04	0.97346	0.97370	2.40E − 04
4.0	0.99910	0.99914	3.80E − 05	0.99808	0.99813	5.30E − 05	0.99629	0.99631	2.20E − 05	0.99322	0.99330	7.70E − 05
5.0	0.99966	0.99967	6.70E − 06	0.99932	0.99937	5.00E − 05	0.99876	0.99883	6.60E − 05	0.99786	0.99783	2.60E − 05
10.0	0.99999	0.99999	1.10E − 06	0.99998	0.99998	1.50E − 06	0.99996	0.99996	6.00E − 07	0.99995	0.99994	2.30E − 06
20.0	1.00000	1.00000	4.80E − 07	1.00000	1.00000	1.20E − 07	1.00000	1.00000	1.20E − 07	1.00000	1.00000	5.40E − 07
30.0	1.00000	1.00000	0.00E + 00	1.00000	1.00000	1.80E − 07	1.00000	1.00000	1.20E − 07	1.00000	1.00000	1.80E − 07
40.0	1.00000	1.00000	0.00E + 00	1.00000	1.00000	6.00E − 08	1.00000	1.00000	0.00E + 00	1.00000	1.00000	0.00E + 00
50.0	1.00000	1.00000	0.00E + 00	1.00000	1.00000	6.00E − 08	1.00000	1.00000	0.00E + 00	1.00000	1.00000	0.00E + 00
75.0	1.00000	1.00000	1.20E − 07	1.00000	1.00000	6.00E − 08	1.00000	1.00000	0.00E + 00	1.00000	1.00000	0.00E + 00
100.0	1.00000	1.00000	1.20E − 07	1.00000	1.00000	6.00E − 08	1.00000	1.00000	0.00E + 00	1.00000	1.00000	0.00E + 00

Note: ERROR = |(Eqn (14) – SIMUL)/Eqn (14)|.

Finally, the cdf is determined via integration of $f_n(t)$ as

$$F_n(t) = \int_0^t f_n(s)ds = \int_0^\infty \frac{1}{v}(1 - e^{-\frac{tv}{\beta}})e^{-nv}\varphi_{m,n}(v)dv. \tag{14}$$

3. Numerical Results

Using Eqs. (12) and (14) present no real difficulties because $\mathrm{Ei}_{m+1}(x)$ can be easily computed via Eq. (9) or by the asymptotic expansion

$$\mathrm{Ei}_m(x) \sim \frac{e^x}{x}\left(1 + \frac{m}{x} + \frac{m(m+1)}{x^2} + \frac{m(m+1)(m+2)}{x^3} + \cdots\right) \quad \text{as } x \to \infty. \tag{15}$$

However, care must be taken to avoid excessive roundoff errors. The author's approach is to compute $e^{-x}\mathrm{Ei}_m(x)$ as a series expansion or asymptotically and then compute $e^{-nv}\varphi_{m,n}(v)$ as

$$e^{-nv}\varphi_{m,n}(v) = (-1)^{n+1}m^n \sum_{r=0}^{\lfloor (n-1)/2 \rfloor} (-\pi^2)^r \binom{n}{2r+1} (e^{-v}\mathrm{Ei}_{m+1}(v))^{n-2r-1} \left(\frac{e^{-v}v^m}{m!}\right)^{2r+1}. \tag{16}$$

As β is a scale parameter, we can, without loss of generality, set $\beta = 1$ and use the pdf

$$f(t) = m(1+t)^{-m-1}, \quad m = 1, 2, \ldots. \tag{17}$$

Figures 1 and 2 display the results of the first five convolutions of the pdf given in Equation (17) with $m = 1$ and $m = 2$ respectively. Tables 1 and 2 display the results of the second through fifth convolutions of the cdf given in Equation (14) along with convolutions obtained via simulation for the case $m = 1$ and $m = 5$, respectively.

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