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DISCUSSIONS AND NOTES

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RATIONAL VALUES OF TRIGONOMETRIC FUNCTIONS

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A recent article* pointed out that cos x is an algebraic number whenever x is equal to an integral number of seconds. An immediate extension of this is that all six standard trigonometric functions have algebraic values for angles measured rationally in degrees. An earlier article† gave a more advanced discussion of these algebraic numbers. It is the purpose of this note to present an elementary proof of the fact, apparently not widely known, that these numbers are scarcely ever rational—in fact only for the very familiar values associated with the sequence $0°, 30°, 45°, \cdots$.

THEOREM. *If x is rational in degrees, then the only possible rational values of the trigonometric functions are: $\sin x, \cos x=0, \pm 1/2, \pm 1$; $\sec x, \csc x=\pm 1, \pm 2$; $\tan x, \cot x=0, \pm 1$.*

The proof can obviously be limited to the cosine and tangent. Assume that cos x is rational but not equal to one of the given values, and that kx equals a multiple of 360°, where k is an integer. Then for any integer n, knx is a multiple of 360°. Now cos knx can be expressed as a polynomial in cos nx with integral coefficients and with leading coefficient equal to 2^{k-1}. Since cos nx is rational and satisfies the equation obtained by equating this polynomial to 1, therefore when cos nx is expressed as a rational fraction in lowest terms, the denominator must be a factor of 2^{k-1}, and incidentally a power of 2. A contradiction is obtained by showing that this denominator of cos nx can be made arbitrarily large. Accordingly, let $\cos \alpha=p/q$, where p is odd and q is a power of 2 greater than the first. Then $\cos 2\alpha=p'/q'$, where $p'=p^2-q^2/2$, an odd number, and $q'=q^2/2$, a power of $2>q$. Thus the terms of the sequence obtained by repeated doubling of the angle, $\cos x$, $\cos 2x$, $\cos 4x, \cdots$, when expressed as rational fractions in lowest terms, have successively larger denominators.

Proof for the tangent follows essentially the same pattern. In this case tan knx can be expressed as the quotient of two polynomials in tan nx, each having integral coefficients, and the numerator having leading coefficient 1 or k. Since tan nx is rational (if finite) and satisfies the equation obtained by equating the numerator polynomial to 0, therefore when tan nx is expressed as a rational

* R. W. Hamming, The transcendental character of cos x, this MONTHLY, vol. 52, 1945, pp. 336–337.

† D. H. Lehmer, A note on trigonometric algebraic numbers, this MONTHLY, vol. 40, 1933, pp. 165–166.

fraction in lowest terms, the denominator must be a factor of k. Let $\tan \alpha = p/q$, where $p \neq \pm q$, $p \neq 0$, $q \neq 0$, and the fraction is in lowest terms. Then $\tan 2\alpha = p'/q'$, where $p' = 2pq \neq 0$ and $q' = q^2 - p^2 \neq 0$. It is impossible for p' to equal $\pm q'$ since the largest possible common factor of p' and q' is 2, this occurring only when p and q are both odd. In any case, when p'/q' is expressed in lowest terms, the resulting fraction is of the same type as p/q, and the new denominator is numerically greater than the first. Again a contradiction is provided by a sequence obtained by repeated doubling.

THE AREA OF A TRIANGLE AS A FUNCTION OF THE SIDES*

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1. Historical remarks. The first mention of the rule giving the area of a triangle as a function of the three sides is found in the works of Heron of Alexandria (1st century). Although it is now believed that this rule pre-dates Heron, demonstrations of it are in his two works, *Metrics* and *Treatise on the Diopter.*

In the book of the three Arabian brothers, Mohammed, Ahmed, and Alhasan, (9th century) we encounter a new demonstration, the first which came to us in Europe. It was reproduced by Leonardo of Pisa in his *Practical Geometry* (1220) and then by Jordanus Nemorarius (13th century), and by most of the geometers of the Renaissance. It is curious that Heron, the Hindus, as well as all the authors we have cited, made an application of this rule to the triangle of sides 13, 14, and 15, whose area is 84. One is led to ask if these three numbers have a common origin, but, as Chasles had observed, the Greeks, the Hindus, and the Arabs may very well have separately become aware of the fact that 13, 14, 15 are the smallest integers which give a rational area for an acute angled triangle.

One finds, still later, other new proofs of the rule by Newton in his *Universal Arithmetic* (1707); by Euler in the *Recent Commentaries of Petersburg* (v. I, 1747, p. 48); by Boscovich in volume V of his *Works* concerning optics and astronomy (1785). This last demonstration is obtained by trigonometric considerations.

2. New demonstration. The author has previously given a very short geometrical demonstration of the formula under consideration (*Mathesis*, 1931, p. 27), and here is another equally simple.

Being given a triangle ABC, ($BC=a$, $CA=b$, $AB=c$, $a+b+c=2p$), let (see figure) B' and B'', C' and C'' be the orthogonal projections of the vertices B and C on the bisectors AD and AD' of angle A. Rectangle $AB''MC'$ has dimensions equal to BB' and CC'', and is the sum of two rectangles, $AB''BB'$ and $B'BMC'$. The first of these rectangles is equal in area to triangle BAE, which has AB' for altitude and $BE=2\ BB'$ for base. The second is equal in area to triangle BEC, which has $NC=B'C'$ for altitude and BE for base. Thus rectangle $AB''MC'$ is equal in area to triangle ABC. Similarly, rectangle $AC''NB'$ is equal in area to triangle ABC, for this rectangle is the difference of rectangles

* Translated from the French by Howard Eves.