On the asymptotic stability of nonnegative matrices in max algebra

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Abstract
In the max algebra system, for an $n \times n$ nonnegative matrix $A = [a_{ij}]$ the eigenequation for max eigenvalue $\lambda$ and corresponding max eigenvector $x$ is $A \otimes x = \lambda x$, where $[A \otimes x]_i = \max_{1 \leq j \leq n} a_{ij} x_j$ and $\mu(A)$ is the maximum circuit geometric mean. It is shown that the following conditions are mutually equivalent: (i) $\eta\|\cdot\|(A) < 1$, for some norm $\|\cdot\|$ on $\mathbb{R}^n$; (ii) $\hat{\eta}(A) < 1$; (iii) $\mu(A) < 1$; (iv) $\lim_{k \to \infty} A_k^{\otimes} = 0$, where $\eta\|\cdot\|(A) = \max_{\|x\|=1, x \geq 0} \|A \otimes x\|$ and $\hat{\eta}(A) = \lim_{k \to \infty} \sup \{ \eta\|\cdot\|(A_k^{\otimes}) \}^\frac{1}{k}$.

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1. Introduction

Following the notation in [4], the max algebra system consists of a set of nonnegative numbers with sum $a \oplus b = \max\{a, b\}$ and the standard product $ab$ for $a, b \geq 0$. For a nonnegative matrix $A = [a_{ij}]$, we may denote $a_{ij}$ by $[A]_{ij}$. Let $\mathbb{R}^{n \times n}$ be the set of all $n \times n$ real matrices and let $A = [a_{ij}]$ and $B = [b_{ij}]$ be nonnegative matrices in $\mathbb{R}^{n \times n}$. The product $A$ and $B$ is denoted by $A \otimes B$, where $[A \otimes B]_{ij} = \max_{1 \leq j \leq n} a_{ij} b_{ij}$.
max_{1 \leq k \leq n} a_{ik} b_{kj}. A^k_{\infty} means A \otimes A, and A^k_{\infty} denotes the kth power of A. Note that A \leq B iff a_{ij} \leq b_{ij} for all 1 \leq i, j \leq n. Let x = [x_i] \in \mathbb{R}^n be a nonnegative vector. The notation A^{k}_x means \text{[A^{k}_x]} = \max_{1 \leq i \leq n} [A^{k}_x]_{x_{j}j}. A^k_x means \{A^k_{\infty} = \sum_{j=1}^n A^k_{\infty} b_{ij}. We say that lim_{k \to \infty} A^{k}_{\infty} = 0 if lim_{k \to \infty} A^{k}_{\infty} b_{ij} = 0, for all 1 \leq i, j \leq n.

Let A = [a_{ij}] be an n \times n nonnegative matrix. The directed graph corresponding to A, denoted by \mathcal{G}(A), is defined by \mathcal{G}(A) = (V, E) with vertex set V = \{1, 2, \ldots, n\} and the set of edges E = \{(i, j) \in V \times V | a_{ij} > 0, 1 \leq i, j \leq n\}. The L(i_0, i_1, i_2, \ldots, i_k) is called a path from vertex i_0 to vertex i_k in \mathcal{G}(A) with length k (k \geq 1) if (i_t, i_{t+1}) \in E for each t = 0, 1, 2, \ldots, k - 1. The weight of a path L(i_1, i_2, \ldots, i_k), denoted by w(L(i_1, i_2, \ldots, i_k)) or simply by w(L), is defined by

\[ w(L(i_1, i_2, \ldots, i_k)) = a_{i_1i_2}a_{i_2i_3} \cdots a_{i_{k-1}i_k}. \]

A circuit of the length k is a path L(i_0, i_1, i_2, \ldots, i_k) with i_k = i_0, where i_0, i_1, \ldots, i_{k-1} are distinct. Associated with this circuit is the circuit geometric mean known as w(L) = (\prod_{t=0}^{k-1} a_{i_ti_{t+1}})^{1/k}. The maximum circuit geometric mean in \mathcal{G}(A) is denoted by \mu(A). Note that we also consider empty circuits, namely, circuits that consist of only one vertex and have length 0. For empty circuits, the associated circuit geometric mean is zero. An n \times n nonnegative matrix A is reducible if there is a permutation matrix P such that

\[ PAP^t = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \]

where A_{11} and A_{22} are square nonnegative submatrices. An n \times n nonnegative matrix A is irreducible if it is not reducible (see, e.g., [10, p. 217]).

Recall that two norms \| \cdot \|_r, \| \cdot \|_r on a vector space are said to be equivalent if whenever a sequence \{x_k\} converges to a vector x with respect to the first norm, then it converges to the same vector with respect to the second norm. It is well known that two norms \| \cdot \|_r, \| \cdot \|_r on a real vector space V are equivalent if and only if there are positive constants m, M such that

\[ m\|x\|_r \leq \|x\|_r \leq M\|x\|_r \]

for all x \in V.

Also, it is known that for finite-dimensional real vector spaces, all norms are equivalent (see, e.g., [9, pp. 272–279]). A norm on \mathbb{R}^{n \times n} is called a generalized matrix norm. A generalized matrix norm \| \cdot \| on \mathbb{R}^{n \times n} is said to be a matrix norm if \| AB \| \leq \| A \| \| B \| for all A, B \in \mathbb{R}^{n \times n}, where AB is the standard matrix product (see, e.g., [9, p. 290]). Since \mathbb{R}^{n \times n} is a finite-dimensional real vector space, all matrix norms on \mathbb{R}^{n \times n} are equivalent. If x = [x_i] \in \mathbb{R}^n, we define \|x\| = \|x_i\| for each x = [x_i], y = [y_i] \in \mathbb{R}^n, we say that \|x\| \leq \|y\| if \|x_i\| \leq \|y_i\| for all i = 1, \ldots, n. A norm \| \cdot \| on \mathbb{R}^n is said to be monotone if \|x\| \leq \|y\| implies \|x\| \leq \|y\| for all x, y \in \mathbb{R}^n. A norm \| \cdot \| on \mathbb{R}^n is said to be absolute if \|x\| = \|\|x\|\| for all x \in \mathbb{R}^n. It is well known that a norm \| \cdot \| on \mathbb{R}^n is monotone if and only if it is absolute (see, e.g., [9, p. 285]).
In the literature, the maximum circuit geometric mean $\mu(A)$ has been studied extensively, and it is known that $\mu(A)$ is a max eigenvalue of $A$. Moreover, if $A$ is irreducible, then $\mu(A)$ is the unique eigenvalue and every eigenvector is positive. Please refer to [1,3,8] for the spectral study. Elsner and Van den Driessche [4–6] provided asymptotic formulas for $\mu(A)$ that involve spectral radii and matrix norms and algorithms of computing $\mu(A)$ and max eigenvector $x$, for an irreducible non-negative matrix $A$ was established as well. Bounds for $\mu(A)$ can be found in [1,2]. The role of $\mu(A)$ plays in the study of powers of a nonnegative matrix $A$ can be found in [4].

2. Results

Let $A$ be an $n \times n$ nonnegative matrix. A scalar $\lambda$ is called a max eigenvalue of $A$ if $A \otimes x = \lambda x$ for some nonnegative vector $x \neq 0$, namely,

$$\max_{1 \leq j \leq n} a_{ij}x_j = \lambda x_i \quad \forall i = 1, 2, \ldots, n.$$ 

The vector $x$ is called a corresponding max eigenvector of $\lambda$. Let $\| \cdot \|$ be any norm on $\mathbb{R}^n$. Associated with this norm $\| \cdot \|$ we define $\eta_{1\|\cdot\|}(A)$ as

$$\eta_{1\|\cdot\|}(A) = \sup_{x \neq 0, x \geq 0} \frac{\| A \otimes x \|}{\| x \|}.$$ 

Observe that for each $\alpha > 0$,

$$\alpha [A \otimes x]_i = \alpha \left( \max_{1 \leq j \leq n} a_{ij}x_j \right) = \max_{1 \leq j \leq n} a_{ij}(\alpha x_j) = [A \otimes (\alpha x)]_i.$$ 

So that $\alpha (A \otimes x) = A \otimes (\alpha x)$. Thus

$$\eta_{1\|\cdot\|}(A) = \sup_{x \neq 0, x \geq 0} \frac{\| A \otimes x \|}{\| x \|} = \sup_{x \neq 0, x \geq 0} \frac{\| A \otimes x \|}{\| x \|} = \sup_{\| x \|=1, x \geq 0} \| A \otimes x \|.$$ 

Since $\| A \otimes x \|$ is a continuous function of $x$ and $\{x : \| x \| = 1, x \geq 0\}$ is a compact set in $\mathbb{R}^n$, we have

$$\eta_{1\|\cdot\|}(A) = \max_{\| x \|=1, x \geq 0} \| A \otimes x \|.$$ 

Let $\| \cdot \|_s$ and $\| \cdot \|_r$ be any two norms on $\mathbb{R}^n$. Since $\mathbb{R}^n$ is a finite-dimensional real vector space, $\| \cdot \|_s$ and $\| \cdot \|_r$ are equivalent. Then there exist $m, M > 0$ such that

$$m\| x \|_s \leq \| x \|_r \leq M\| x \|_s$$ 

for all $x \in \mathbb{R}^n$. 


so that for each $n \times n$ nonnegative matrix $A$, we have
\[
\eta_{1,1}(A) = \max_{x \neq 0, x \geq 0} \frac{\|A \otimes x\|_r}{\|x\|_r} \leq \max_{x \neq 0, x \geq 0} \frac{M}{m} \frac{\|A \otimes x\|_s}{\|x\|_s} = \frac{M}{m} \eta_{1,1}(A).
\]
Hence there is a positive integer $L$ such that
\[
\left[ \frac{1}{L} \eta_{1,1}(A^k) \right]^\frac{1}{k} \leq [\eta_{1,1}(A^k)]^\frac{1}{k} \leq [L\eta_{1,1}(A^k)]^\frac{1}{k} \quad \forall k = 1, 2, \ldots.
\]
It follows
\[
\limsup_{k \to \infty} [\eta_{1,1}(A^k)]^\frac{1}{k} = \limsup_{k \to \infty} [\eta_{1,1}(A^k)]^\frac{1}{k},
\]
so that $\limsup_{k \to \infty} [\eta_{1,1}(A^k)]^\frac{1}{k}$ does not depend on the particular choice for the norm on $\mathbb{R}^n$. Now we define $\hat{\eta}(A)$ by
\[
\hat{\eta}(A) = \limsup_{k \to \infty} [\eta_{1,1}(A^k)]^\frac{1}{k}.
\]
From the proof 1 of Theorem 2 and the proof of Lemma 1 [1], we have concluded the following result. For easy reference, we state this as a theorem.

**Theorem 1.** Let $A$ be an $n \times n$ nonnegative matrix. Then the maximum circuit geometric mean $\mu(A)$ is the largest max eigenvalue of $A$.

The following four lemmas are needed for the proof of the main Theorem 2.

**Lemma 1.** Let $A, B$ be $n \times n$ nonnegative matrices and $\| \cdot \|$ be a norm on $\mathbb{R}^n$. Then

(i) $\eta_{1,1}(A) = 0 \iff A = 0$.
(ii) $\eta_{1,1}(A \otimes B) \leq \eta_{1,1}(A) \eta_{1,1}(B)$.
(iii) $\mu(A) \leq \eta_{1,1}(A)$.

**Proof.** (i) Let $e_i$ be the vector on $\mathbb{R}^n$ whose $i$th component is 1 and 0 otherwise. If $\eta_{1,1}(A) = 0$ then for each $i$ we have
\[
\left\| A \otimes \frac{e_i}{\|e_i\|} \right\| \leq \eta_{1,1}(A) = 0.
\]
This implies that $a_{ji} = 0$ for all $j = 1, 2, \ldots, n$. Thus $A = 0$. The converse is clear.

(ii) It follows that
\[
\begin{align*}
\eta_{1,1}(A \otimes B) &= \max_{\|x\|_{1, x \geq 0}} \frac{\|A \otimes B \otimes x\|}{\|x\|_r} \\
&\leq \max_{\|x\|_{1, x \geq 0}} (\eta_{1,1}(A) \|B \otimes x\|) \\
&= \eta_{1,1}(A) \left( \max_{\|x\|_{1, x \geq 0}} \|B \otimes x\| \right) \\
&= \eta_{1,1}(A) \eta_{1,1}(B).
\end{align*}
\]
(iii) By Theorem 1, there is a nonnegative vector \( \hat{x} \neq 0 \in \mathbb{R}^n \) such that \( A \otimes \hat{x} = \mu(A) \hat{x} \). Hence
\[
\eta_1(A) = \max_{\|x\| \neq 0, x \geq 0} \frac{\|A \otimes x\|}{\|x\|} \geq \frac{\|A \otimes \hat{x}\|}{\|\hat{x}\|} = \frac{\|\mu(A) \hat{x}\|}{\|\hat{x}\|} = \mu(A). \tag*{□}
\]

**Lemma 2.** Let \( A \) be an \( n \times n \) nonnegative matrix and \( \| \cdot \| \) be a norm on \( \mathbb{R}^n \). Then \( \hat{\eta}(A) \leq [\eta_{\|\cdot\|}(A^k)]^2 \), \( k = 1, 2, \ldots \).

**Proof.** By Lemma 1(ii), we have
\[
\eta_1(A^{r+j}) = \eta_1(A^r \otimes A^j) \leq \eta_{\|\cdot\|}(A^r) \eta_1(A^j).
\]
Fix \( k \) and let \( l \geq k \). Write \( l = mk + j \) with \( 0 \leq j \leq k - 1 \). Then
\[
\eta_1(A^l) = \eta_{\|\cdot\|}(A^mk+j) \\
\leq \eta_{\|\cdot\|}(A^mk) \eta_{\|\cdot\|}(A^j) \\
\leq [\eta_{\|\cdot\|}(A^k)]^2 \eta_{\|\cdot\|}(A^j) \\
\leq [\eta_{\|\cdot\|}(A^k)]^2 \eta_{\|\cdot\|}(A^j).
\]
So that
\[
[\eta_{\|\cdot\|}(A^l)]^2 \leq [\eta_{\|\cdot\|}(A^k)]^2 \eta_{\|\cdot\|}(A^j) \\
= [\eta_{\|\cdot\|}(A^k)]^2 \eta_{\|\cdot\|}(A^j) \\
\leq [\eta_{\|\cdot\|}(A^k)]^2 \eta_{\|\cdot\|}(A^j) \\
= [\eta_{\|\cdot\|}(A^k)]^2.
\]
Taking \( l \to \infty \), one easily obtains
\[
\hat{\eta}(A) \leq [\eta_{\|\cdot\|}(A^k)]^2.
\]
This completes the proof. \( \square \)

By Lemma 2, we have \( \hat{\eta}(A) = \lim_{k \to \infty} [\eta_{\|\cdot\|}(A^k)]^2 \).

**Lemma 3.** Let \( A \) be an \( n \times n \) nonnegative matrix with \( \mu(A) \leq 1 \). Then there is a constant \( M \geq 1 \) such that for each \( l > n \),
\[
[A^k]_{ij} \leq M^n \mu(A)^{l-n}
\]
for all \( 1 \leq i, j \leq n \).

**Proof.** Let \( \alpha = \max(a_{ij} : 1 \leq i, j \leq n) \). Put \( M = \alpha \) if \( \alpha \geq 1 \), and \( M = 1 \) if \( \alpha < 1 \). Let \( i, j \) be fixed and let \( L \) be an \( l \) path from vertex \( i \) to vertex \( j \) in the digraph \( \mathcal{D}(A) \) with \( w(L) = [A^k]_{ij} \). Since \( l > n \), path \( L \) contains a circuit, moreover, at least \( l - n \) its vertices are on a circuit. Let these circuits be \( C_1, \ldots, C_k \) with length \( l_1, \ldots, l_k \),
hence \(l_1 + \cdots + l_k \geq l - n\). The remaining vertices, so at most \(n\) vertices, form a path \(L'\) with weight less than or equal to \(M^n\). Then
\[
\begin{align*}
[A^l_i]_{ij} &= w(L) \\
&= w(C_1) \cdots w(C_k) w(L') \\
&=[\hat{w}(C_1)]^l_{ij} \cdots [\hat{w}(C_k)]^l_{ij} w(L') \\
&\leq [\mu(A)]^l_{ij} \cdots [\mu(A)]^l_{ij} M^n \\
&\leq [\mu(A)]^{l-n} M^n \text{ (by } \mu(A) \leq 1\).
\end{align*}
\]
This completes the proof. \(\Box\)

**Lemma 4.** Let \(A\) be an \(n \times n\) nonnegative matrix and \(\| \cdot \|\) be a monotone norm on \(\mathbb{R}^n\). Then for each positive integer \(k\), and positive numbers \(a_0, a_1, \ldots, a_k\), the function
\[
\|x\|_s = a_0 \| |x| \| + a_1 \| A \otimes |x| \| + a_2 \| A^2 \otimes |x| \| + \cdots + a_k \| A^k \otimes |x| \|, \quad x \in \mathbb{R}^n
\]
is a monotone norm on \(\mathbb{R}^n\).

**Proof.** It is clear that \(\|x\|_s = 0 \iff x = 0\). Let \(\alpha \in \mathbb{R}\) be given. Then
\[
\|\alpha x\|_s = a_0 \| |\alpha x| \| + a_1 \| A \otimes |\alpha x| \| + a_2 \| A^2 \otimes |\alpha x| \| + \cdots + a_k \| A^k \otimes |\alpha x| \|
= a_0 \| |x| \| + |\alpha| a_1 \| A \otimes |x| \| + |\alpha| a_2 \| A^2 \otimes |x| \|
+ \cdots + |\alpha| a_k \| A^k \otimes |x| \|
= |\alpha| \|x\|_s.
\]
Observe that for each \(n \times n\) nonnegative matrix \(B\) and \(x, y \in \mathbb{R}^n\), we have
\[
[B \otimes |x + y|]_i = \max_{1 \leq j \leq n} b_{ij}|x_j + y_j|
\leq \max_{1 \leq j \leq n} (b_{ij}|x_j| + b_{ij}|y_j|)
\leq \max_{1 \leq j \leq n} b_{ij}|x_j| + \max_{1 \leq j \leq n} b_{ij}|y_j|
= [B \otimes |x|]_i + [B \otimes |y|]_i.
\]
This implies that
\[
B \otimes |x + y| \leq B \otimes |x| + B \otimes |y|.
\]
Since \(\| \cdot \|\) is a monotone norm, we have
\[
\|B \otimes |x + y|\| \leq \|B \otimes |x|\| + \|B \otimes |y|\| \leq \|B \otimes |x|\| + \|B \otimes |y|\|.
\]
Now we claim that \( \|x + y\|_s \leq \|x\|_s + \|y\|_s \).

\[
\|x + y\|_s = a_0(\|x + y\| + a_1\|A \otimes |x + y|\| + a_2\|A^2 \otimes |x + y|\|) + \cdots + a_k(\|A^k \otimes |x + y|\|)
\leq a_0(\|x\| + \|y\|) + a_1(\|A \otimes |x|\| + \|A \otimes |y|\|) + a_2(\|A^2 \otimes |x|\| + \|A^2 \otimes |y|\|)
+ \cdots + a_k(\|A^k \otimes |x|\| + \|A^k \otimes |y|\|)
= \|x\|_s + \|y\|_s.
\]

It is clear that \( \|x\|_s = \|\cdot\|_s \). Thus \( \| \cdot \|_s \) is an absolute norm, and hence a monotone norm. \( \square \)

**Theorem 2.** Let \( A \) be an \( n \times n \) nonnegative matrix. Then the following statements are mutually equivalent.

(i) \( \eta_\| \frac{\|A\|}{\|\cdot\|} < 1 \) for some norm \( \| \cdot \| \) on \( \mathbb{R}^n \).

(ii) \( \hat{\eta}(A) < 1 \).

(iii) \( \mu(A) < 1 \).

(iv) \( \lim_{k \to \infty} A^k = 0 \).

**Proof.** (i)\( \Rightarrow \) (ii). By Lemma 2, we have \( \hat{\eta}(A) \leq \eta_\| \frac{\|A\|}{\|\cdot\|} < 1 \).

(ii)\( \Rightarrow \) (iii). By Lemma 1(iii), for each \( k \),

\[
\eta_\| \frac{\|A^k\|}{\|\cdot\|} \geq \mu(A^k).
\]

Since \( \mu(A^k) = [\mu(A)]^k \) (see [4, p. 29]), we have \( \eta_\| \frac{\|A^k\|}{\|\cdot\|} \geq [\mu(A)]^k \). Thus \( [\eta_\| \frac{\|A^k\|}{\|\cdot\|}] \geq \mu(A) \), and hence \( \hat{\eta}(A) \geq \mu(A) \).

(iii)\( \Rightarrow \) (iv). By Lemma 3, there is a constant \( M \) such that for each \( k > n \),

\[
[\|A^k\|]_{ij} \leq M^k [\mu(A)]^{k-n}
\]

for all \( 1 \leq i, j \leq n \). Since \( \mu(A) < 1 \), we have \( \lim_{k \to \infty} A^k = 0 \).

(iv)\( \Rightarrow \) (i). Since \( \lim_{k \to \infty} A^k = 0 \), we have \( \lim_{k \to \infty} [A^k]_{ij} = 0 \) for all \( 1 \leq i, j \leq n \). Then there is a positive integer \( N \) such that \( [A^N]_{ij} < 1 \) for all \( 1 \leq i, j \leq n \). Let

\[
\|x\|_s = \|x\|_\infty + \|A \otimes |x|\|_\infty + \|A^2 \otimes |x|\|_\infty + \cdots + \|A^{N-1} \otimes |x|\|_\infty.
\]

By Lemma 4, \( \| \cdot \|_s \) is a monotone norm on \( \mathbb{R}^n \). Observe that

\[
[|A \otimes x|]_i = \max_{1 \leq j \leq n} a_{ij} |x_j| \leq \max_{1 \leq j \leq n} a_{ij} |x_j| = [A \otimes |x|]_i.
\]

Thus \( |A \otimes x| \leq A \otimes |x| \), and hence we obtain that

\[
\|B \otimes |A \otimes x|\|_\infty \leq \|B \otimes A \otimes |x|\|_\infty
\]
for all nonnegative matrices $B$. So that

$$
\|A \otimes x\|_s = \|A \otimes |x|\|_\infty + \|A \otimes |x|\|_\infty + \|A \otimes |x|\|_\infty + \cdots + \|A \otimes |x|\|_\infty 
\leq \|A \otimes |x|\|_\infty + \|A \otimes |x|\|_\infty + \|A \otimes |x|\|_\infty + \cdots + \|A \otimes |x|\|_\infty 
= \max_{\|x\|_\infty = 1, x \geq 0} \|A \otimes |x|\|_\infty
\leq \|A \otimes |x|\|_\infty + \|A \otimes |x|\|_\infty + \cdots + \|A \otimes |x|\|_\infty + \|x\|_\infty
= \|x\|_s.
$$

This implies that $\eta_B(A) < 1$. □

Example 1. Consider the following $2 \times 2$ nonnegative matrix

$$
A = \begin{bmatrix}
1 & 1 \\
1/2 & 1/2
\end{bmatrix}.
$$

We have $\mu(A) = 1/2$ and $\rho(A) = 1$, where $\rho(A)$ is the standard spectral radius of $A$. Then for any matrix norm $\| \cdot \|$, we have

$$
\|A\| \geq \rho(A) = 1
$$

(see, e.g., [9, p. 297]). Moreover,

$$
\eta_{\| \cdot \|_\infty}(A) = \max_{\|x\|_\infty = 1, x \geq 0} \|A \otimes x\|_\infty
= \max_{\|x\|_\infty = 1, x \geq 0} \\left\| \begin{bmatrix}
1/2x_1 + 1/2x_2 \\
1/2x_1 + 1/2x_2
\end{bmatrix}\right\|_\infty
= \frac{1}{2} = \mu(A) < 1
$$

and $\lim_{k \to \infty} A_k^\otimes = 0$.

We establish the max algebra version of the Gelfand spectral radius formula as follows.

Theorem 3. Let $A$ be an $n \times n$ nonnegative matrix. Then $\hat{\eta}(A) = \mu(A)$.

Proof. Observe that for each $\gamma > 0$, $\left(\frac{A}{\gamma}\right)^k = \frac{1}{\gamma^k} A_k^\otimes$ for all $k = 1, 2, \ldots$ Therefore

$$
\hat{\eta}\left(\frac{A}{\gamma}\right) = \lim_{k \to \infty} \left[ \eta_{\| \cdot \|} \left(\frac{A}{\gamma}\right)^k \right]^{1/2}
= \lim_{k \to \infty} \left[ \eta_{\| \cdot \|} \left(\frac{1}{\gamma^k} A_k^\otimes\right) \right]^{1/2}
$$
\[
\lim_{k \to \infty} \left[ \max_{\|x\|_1 = 1, x \geq 0} \left\| \left( \frac{1}{\gamma^k} A_k^k \right) \otimes x \right\| \right]^{\frac{1}{k}} = \frac{1}{\gamma} \lim_{k \to \infty} \left[ \max_{\|x\|_1 = 1, x \geq 0} \left\| A_k^k \otimes x \right\| \right]^{\frac{1}{k}} = \frac{1}{\gamma} \hat{\eta}(A).
\]

Note that \( \mu \left( \frac{A}{\gamma} \right) = \frac{1}{\gamma} \mu(A) \). Thus

\[
\hat{\eta}(A) < \gamma \iff \hat{\eta} \left( \frac{A}{\gamma} \right) < 1 \iff \mu \left( \frac{A}{\gamma} \right) < 1 \text{ (by Theorem 2)} \iff \mu(A) < \gamma.
\]

Hence \( \hat{\eta}(A) = \mu(A) \).

This paper provides an alternative proof using the notion \( \eta_{1,1}(A) \) for an asymptotic formula which was proposed by Elsner and Van den Driessche’s lemma [4].

**Theorem 4.** Let \( A \) be an \( n \times n \) nonnegative matrix and \( \| \cdot \| \) be any matrix norm. Then \( \mu(A) = \lim_{k \to \infty} \| A_k^k \|^{\frac{1}{k}} \).

**Proof.** Since \( A \otimes x \leq Ax \leq nA \otimes x \) for all \( x \geq 0 \) and \( \| \cdot \|_\infty \) is a monotone norm, we have

\[
\lim_{k \to \infty} \left( \max_{\|x\|_\infty = 1, x \geq 0} \| A_k^k \otimes x \|_\infty \right)^{\frac{1}{k}} \leq \lim_{k \to \infty} \left( \max_{\|x\|_\infty = 1, x \geq 0} \| A_k^k \otimes x \| \right) \leq \lim_{k \to \infty} \left( \max_{\|x\|_\infty = 1, x \geq 0} n \| A_k^k \otimes x \|_\infty \right)^{\frac{1}{k}}.
\]

Hence

\[
\hat{\eta}(A) = \lim_{k \to \infty} \left( \max_{\|x\|_\infty = 1, x \geq 0} \| A_k^k \otimes x \|_\infty \right)^{\frac{1}{k}} \leq \lim_{k \to \infty} \left( \max_{\|x\|_\infty = 1, x \geq 0} \| A_k^k \otimes x \|_\infty \right)^{\frac{1}{k}}
\]

□
and

$$\lim_{k \to \infty} \left( \max_{\|x\|_\infty = 1, x \geq 0} \|A_k \otimes x\|_\infty \right)^{\frac{1}{k}} \leq \lim_{k \to \infty} \left( \max_{\|x\|_\infty = 1, x \geq 0} \|nA_k \otimes x\|_\infty \right)^{\frac{1}{k}}$$

$$= \lim_{k \to \infty} n^{\frac{1}{k}} \left( \max_{\|x\|_\infty = 1, x \geq 0} \|A_k \otimes x\|_\infty \right)^{\frac{1}{k}}$$

$$= \lim_{k \to \infty} \left( \max_{\|x\|_\infty = 1, x \geq 0} \|A_k \otimes x\|_\infty \right)^{\frac{1}{k}}$$

$$= \hat{\eta}(A).$$

So that

$$\hat{\eta}(A) = \lim_{k \to \infty} \left( \max_{\|x\|_\infty = 1, x \geq 0} \|A_k \otimes x\|_\infty \right)^{\frac{1}{k}}. \quad (1)$$

For each nonnegative matrix $B$, we have

$$\max_{\|x\|_\infty = 1, x \geq 0} \|Bx\|_\infty \leq \max_{\|x\|_\infty = 1} \|Bx\|_\infty \leq \max_{\|x\|_\infty = 1} \|B\|_\infty \|x\|_\infty$$

$$= \max_{\|x\|_\infty = 1, x \geq 0} \|Bx\|_\infty.$$

Therefore,

$$\max_{\|x\|_\infty = 1} \|Bx\|_\infty = \max_{\|x\|_\infty = 1} \|B\|_\infty \|x\|_\infty = \max_{\|x\|_\infty = 1, x \geq 0} \|Bx\|_\infty. \quad (2)$$

By Theorem 3 and Eqs. (1) and (2), we have

$$\mu(A) = \hat{\eta}(A)$$

$$= \lim_{k \to \infty} \left( \max_{\|x\|_\infty = 1, x \geq 0} \|A_k \otimes x\|_\infty \right)^{\frac{1}{k}}$$

$$= \lim_{k \to \infty} \left( \max_{\|x\|_\infty = 1} \|A_k \otimes x\|_\infty \right)^{\frac{1}{k}}$$

$$= \lim_{k \to \infty} \|A_k \otimes x\|_\infty^{\frac{1}{k}}.$$

Since any two norms on a finite dimensional space are equivalent, there are $m, M > 0$ such that

$$m\|A_k\| \leq \|A_k \otimes x\|_\infty \leq M\|A_k\|.$$

Thus

$$\lim_{k \to \infty} \|A_k \otimes x\|_\infty^{\frac{1}{k}} = \lim_{k \to \infty} \|A_k\|^{\frac{1}{k}}$$

and hence $\mu(A) = \lim_{k \to \infty} \|A_k\|^{\frac{1}{k}}$. \qed
The following theorem is well known (see [1,7]), we give an alternative proof.

**Theorem 5.** Let $A$ be an $n \times n$ nonnegative matrix. Then

$$
\mu(A) \leq \rho(A) \leq n \mu(A).
$$

**Proof.** By Theorem 4,

$$
\mu(A) = \lim_{k \to \infty} \|A_k\|_{\infty}^{\frac{1}{k}}
\leq \lim_{k \to \infty} \|A\|_{\infty}^{\frac{1}{k}} \quad \text{(by $A_k \leq A$)}
= \rho(A).
$$

On the other hand,

$$
n \mu(A) = n \hat{\eta}(A)
= n \lim_{k \to \infty} \left( \max_{\|x\|_{\infty} = 1} \|A_k \otimes x\|_{\infty} \right)^{\frac{1}{k}}
= n \lim_{k \to \infty} \left( \max_{\|x\|_{\infty} = 1} \|A_k \otimes x\|_{\infty} \right)^{\frac{1}{k}}
= \lim_{k \to \infty} \left( \max_{\|x\|_{\infty} = 1} \|n A_k \otimes x\|_{\infty} \right)^{\frac{1}{k}}
\geq \lim_{k \to \infty} \left( \max_{\|x\|_{\infty} = 1} \|n A_k \otimes x\|_{\infty} \right)^{\frac{1}{k}} \quad \text{(by $n A_k \otimes x \geq A_k |x|$)}
\geq \lim_{k \to \infty} \left( \max_{\|x\|_{\infty} = 1} \|A |x\|_{\infty} \right)^{\frac{1}{k}} \quad \text{(by $n A_k \otimes x \geq A_k$)}
= \lim_{k \to \infty} \left( \max_{\|x\|_{\infty} = 1} \|A |x\|_{\infty} \right)^{\frac{1}{k}} \quad \text{(by (2))}
= \rho(A).
$$

Thus

$$
\mu(A) \leq \rho(A) \leq n \mu(A). \quad \square
$$

**Theorem 6.** Let $A$ be an $n \times n$ nonnegative matrix. Then for any $\epsilon > 0$ there is a norm $\|\cdot\|$ on $\mathbb{R}^n$ such that $\eta_{\|\cdot\|}(A) \leq \hat{\eta}(A) + \epsilon$.

**Proof.** Let $\epsilon > 0$ be given. Put $\alpha = \hat{\eta}(A) + \epsilon$. Choose $N$ so that

$$
\eta_{\|\cdot\|}(A_N) \leq \alpha^N.
$$
Define $\| \cdot \|_s$ by
\[
\|x\|_s = \|x\|_\infty + \frac{1}{\alpha} \|A \otimes |x|\|_\infty + \frac{1}{\alpha^2} \|A^2 \otimes |x|\|_\infty + \cdots + \frac{1}{\alpha^{N-1}} \|A^{N-1} \otimes |x|\|_\infty.
\]
By Lemma 4, $\| \cdot \|_s$ is a monotone norm. Since $|A \otimes x| \leq A \otimes |x|$, we have
\[
\|A \otimes x\|_s = \|A \otimes x\|_\infty + \frac{1}{\alpha} \|A \otimes A \otimes x\|_\infty + \frac{1}{\alpha^2} \|A^2 \otimes A \otimes x\|_\infty + \cdots + \frac{1}{\alpha^{N-1}} \|A^{N-1} \otimes A \otimes x\|_\infty
\]
\[
\leq \|A \otimes |x|\|_\infty + \frac{1}{\alpha} \|A^2 \otimes |x|\|_\infty + \cdots + \frac{1}{\alpha^{N-1}} \|A^{N-1} \otimes |x|\|_\infty
\]
\[
= \alpha \left( \frac{1}{\alpha} \|A \otimes |x|\|_\infty + \frac{1}{\alpha^2} \|A^2 \otimes |x|\|_\infty + \cdots + \frac{1}{\alpha^{N-1}} \|A^{N-1} \otimes |x|\|_\infty \right)
\]
\[
\leq \alpha \left( \frac{1}{\alpha} \|A \otimes |x|\|_\infty + \frac{1}{\alpha^2} \|A^2 \otimes |x|\|_\infty + \cdots + \frac{1}{\alpha^{N-1}} \|A^{N-1} \otimes |x|\|_\infty \right)
\]
\[
= \alpha \|x\|_s.
\]
Thus
\[
\eta_{\| \cdot \|_s}(A) \leq \alpha \hat{\eta}(A) + \epsilon.
\]  
\[\square\]

**Corollary 1.** Let $A$ be an $n \times n$ nonnegative matrix. Then for any $\epsilon > 0$ there is a norm $\| \cdot \|$ on $\mathbb{R}^n$ such that $\mu(A) \leq \eta_{\| \cdot \|}(A) \leq \mu(A) + \epsilon$.

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**References**
