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THE DETERMINANT OF THE ADJACENCY MATRIX OF A GRAPH[1, 2]

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THERE ARE THREE MUTUALLY INTERCHANGEABLE WAYS of representing a binary relation R on a finite set S. We may list the ordered pairs of objects in the relation, we may draw a directed graph (or more briefly, a *digraph*) D whose points are the objects in S and in which we draw a directed line from one point to another whenever these two points form one of the ordered pairs in R, and finally we may write a square matrix $A = A(D)$ in which the entry in the i, j cell is 1 if there is a line of D from the i'th point to the j'th point, and this entry is 0 otherwise. The matrix A is known as the *adjacency matrix* of the digraph D.

It is sometimes customary when speaking of digraphs to restrict consideration to irreflexive relations. Thus, in this special case, there is no directed line in D from any point to itself, and in A every diagonal entry is 0. Our object is to obtain a formula for the determinant of A in terms of the structural properties of D. We will also find a formula for the determinant of the adjacency matrix of an ordinary graph (or more briefly, a *graph*) which corresponds to an irreflexive symmetric relation. The extension to arbitrary relations, which are not necessarily irreflexive, is straightforward.

In their classical book of problems, Pólya and Szegö [6] proposed the special cases of finding the determinant of the adjacency matrix of the tetrahedron (-3), hexahedron (9), and octahedron (0), as Exercise 1 in their chapter on determinants and quadratic forms. Collatz and Sinogowitz [2] have studied the properties of the eigenvalues of the adjacency matrix of an ordinary graph and discussed the value of the determinant of A while describing the coefficients of the polynomial $| A - \lambda I |$. Thus their work contains our equation (7) implicitly. Our formulas (2) and (7) give explicit expressions for the value of the product of the eigenvalues of A for digraphs and ordinary graphs respectively. These follow from more general formulas (1) and (5) which give a structural interpretation to the determinant of the matrix obtained from A when a variable corresponding to a line replaces each entry of value 1.

Since formula (1) also gives the determinant of any square matrix, including those with nonzero diagonal entries, it can be regarded as a combinatorial formula for evaluating determinants and can in fact be taken as an alternate definition of a determinant.

Two graphs G_1 and G_2 are *isomorphic* if there is a $1 - 1$ correspondence between their sets of points which preserves adjacency. Thus G_1 and G_2 are isomorphic if and only if their adjacency matrices A_1 and A_2 have the property that for some permutation matrix P, $A_2 = PA_1P^{-1}$.

[1] This note was suggested by Professor Pólya, to whom the author is grateful for his comments. It was written during a visit at the Numerical Analysis Group, Stanford University in September 1960. Presented to the Amer. Math. Soc. April 6, 1961.

[2] Received by the editors July 21, 1961 and in revised form February 20, 1962.

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Conjecture and Question

During the ten minute talk mentioned in footnote 1, I made the following statement as a conjecture.

Conjecture: Two graphs G_1 and G_2 are isomorphic if their adjacency matrices A_1 and A_2 have the same eigenvalue spectra.

R. C. Bose, who was present, immediately announced that the conjecture was not true, and later provided a counterexample consisting of two nonisomorphic graphs with 16 points each, whose adjacency matrices have the same eigenvalues. Subsequently, other counterexamples with 16 points have been found by R. H. Bruck and A. J. Hoffman. Several disproofs of this conjecture will soon appear in the literature. W. T. Tutte (oral communication) has suggested that the conjecture be withdrawn, and this is hereby done. However, the following question remains.

Question: What is the smallest number of points in two nonisomorphic graphs G_1 and G_2 which serve as a counterexample?

It has been verified by exhaustive methods that the conjecture holds for all graphs with up to 6 points. It is suspected that the answer to the question is 16.

Digraphs

Let D be a digraph whose points are $v_1, v_2, \cdots, v_p$ and whose directed lines are $x_1, x_2, \cdots, x_q$. By an abuse of notation which will be clear by context, we refer to x_i both as the i'th line and as a variable associated with this line. There is an additional matrix which can be defined for the digraph D which will be called the *variable adjacency matrix*. This matrix is denoted $A(D, x)$ and is constructed as follows: the i, j entry is x_k if and only if there is a line in D from v_i to v_j and x_k is this line, and this entry is 0 if there is no such line in D. Let $A(D)$ be the adjacency matrix of D. Since $A(D)$ is obtained from $A(D, x)$ by substituting $x_k = 1$ for each of the variables standing for the lines of D, it follows that the value of the determinant $|A(D)|$ is immediately determined from that of $|A(D, x)|$ by substituting $x_k = 1$ for each line. We will call $|A(D)|$ the *determinant* of D and $|A(D, x)|$ the *variable determinant* of D.

The entries in the adjacency matrix $A = A(D)$ of digraph D clearly depend on the ordering of the points. But the value of the determinant $|A|$ is independent of this ordering. For the adjacency matrix with any other ordering is of the form PAP^{-1} for some permutation matrix P, and $|PAP^{-1}| = |P| \cdot |A| \cdot |P^{-1}| = |A|$.

We illustrate with the digraph D of Fig. 1, which has five points and ten lines. The variable adjacency matrix of this digraph is

$$A(D, x) = \begin{bmatrix} 0 & x_1 & 0 & 0 & x_6 \\ 0 & 0 & x_2 & x_7 & 0 \\ 0 & x_9 & 0 & x_3 & 0 \\ 0 & x_{10} & x_8 & 0 & x_4 \\ x_5 & 0 & 0 & 0 & 0 \end{bmatrix}$$

so that its adjacency matrix is

$$A(D) = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

One may readily verify that

$$| A(D, x) | = x_1x_2x_3x_4x_5 - x_5x_6(x_2x_3x_{10} + x_7x_8x_9)$$

from which we see at once that

$$| A(D) | = -1.$$

In Fig. 2 we show the subgraphs of D corresponding to the three terms in the expression for $| A(D, x) |$.

This example has been developed in detail in order to bring out the structure of the nonvanishing terms in the polynomial which is obtained by expanding the determinant of the variable adjacency matrix of a digraph. A general observation on this subject has been made implicitly in a previous article [4], and we only state the result here.

THEOREM A. *A term in the expansion of the determinant of a digraph is nonzero if and only if the lines of a digraph corresponding to the entries in this term constitute*

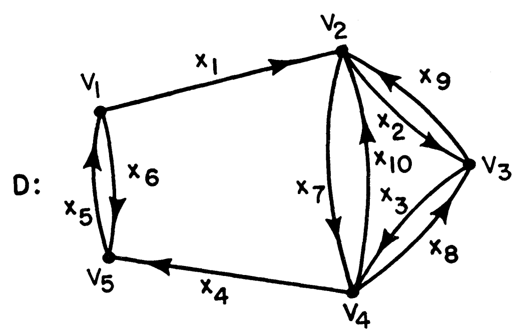

FIGURE 1

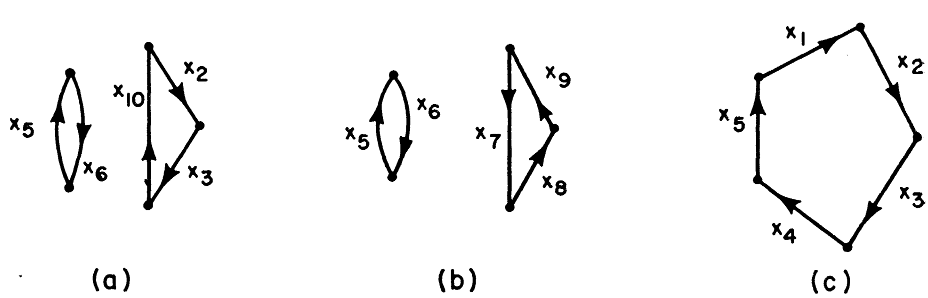

FIGURE 2

a collection of directed cycles such that each point of D is contained in exactly one of these cycles.

It is easy to build on this result to obtain an explicit formula for the determinant $|A(D, x)|$, which we shall state as Theorem 1. The *indegree* (*outdegree*) of a point v of a digraph D is the number of lines to v (from v). By a *linear subgraph* of a digraph we will mean a subgraph of the kind described in Theorem A, that is, a subgraph of D in which each point of D has indegree 1 and also outdegree 1. Thus, a linear subgraph of a digraph D consists of a collection of directed cycles such that each point occurs in exactly one cycle. This has been noted in König [3].

Let n be the number of distinct linear subgraphs of D. Let D_i be the i'th linear subgraph. Let $p_i(x)$ be the product of the variables x_k for the lines in D_i. Let e_i be the number of even cycles in D_i.

THEOREM 1. *The variable determinant of a digraph is given by the following formula*:

$$|A(D,x)| = \sum_{i=1}^{n} (-1)^{e_i} p_i(x). \tag{1}$$

PROOF. The factor $p_i(x)$ in each term of the sum which gives the value of this determinant is an immediate consequence of Theorem A. It remains only to justify the factor $(-1)^{e_i}$. The reason for this may be seen by referring to Fig. 2. If we consider the cycle of length 2 in Fig. 2a as a digraph, then obviously its variable determinant is $-x_5x_6$. If we take the directed cycle of length 3 in Fig. 2a as a digraph in its own right, then it is immediately evident that the corresponding determinant is $x_2x_3x_{10}$. Continuing, we see that any directed cycle of even length, considered as a digraph, will have for its variable determinant the negative of the product of its lines, while any odd cycle will have simply the product of its lines. Since the determinant $|A(D, x)|$ is independent of the ordering of the points of D, we may select a separate ordering for the points in each linear subgraph of D, so that its adjacency matrix is decomposed into diagonal submatrices. But since the determinant of a matrix which is decomposed into diagonal submatrices is the product of the determinants of these submatrices, it follows that the variable determinant of a linear subgraph of D, ignoring the remaining lines of D, is the product of the corresponding determinants for each of the directed cycles which together constitute this linear subgraph, proving that $|A(D_i, x)| = (-1)^{e_i}p_i(x)$. To complete the proof of the theorem, it only needs to be noted that

$$|A(D,x)| = \sum_{i=1}^{n} |A(D_i, x)|. \tag{2}$$

But this is an immediate consequence of the definition of a determinant.

We immediately determine the determinant of the adjacency matrix of a digraph by setting each $x_k = 1$,

$$|A(D)| = \sum_{i=1}^{n} (-1)^{e_i}. \tag{3}$$

A *path in a digraph* is an alternating sequence of distinct points and directed lines beginning and ending with a point such that each line is preceded by its first point and followed by its second point. If there is a path from v_1 to v_2 in D, we say that v_2 is *reachable from* v_1. A digraph is *strongly connected* or *strong* if any two points are mutually reachable. A *strong component* of a digraph is a maximal strong subgraph. Thus a strong component C of D is a subgraph of D which is itself a strong digraph and any other subgraph of D which contains C properly is not strong.

THEOREM 2. *The determinant of the adjacency matrix of a digraph is the product of the corresponding determinants of its strong components.*

PROOF. In the formula of Theorem 1, we combined the observations that the variable determinant of a digraph is the sum of the corresponding determinants of its linear subgraphs, and that the variable determinant of a linear subgraph is the product of the variable determinants of its directed cycles. Since the points and lines of a directed cycle necessarily lie in the same strong component, it follows that the product of the variable determinants of each of the strong components of a digraph gives the variable determinant for the entire digraph.

This result is stated in the following equation, in which $C_1, C_2, \cdots, C_s$ denote the strong components.

$$|A(D, x)| = \prod_{i=1}^{s} |A(C_i, x)|. \tag{4}$$

In the case of a digraph with several strong components, this formula would simplify the calculations required to find the determinant of its adjacency matrix. A line of a digraph D which does not lie in any strong component cannot be in any cycle and therefore is never contained in a linear subgraph of D. Thus, to find the determinant of D it is sufficient to find the determinant of that subgraph of D obtained by deleting all those lines that are not contained in any strong component. In the article [4], there is an algorithm for finding the strong components of a digraph. Using this method, the lines to be deleted are readily determined. After removing these lines, the digraph (if it is not strongly connected to start) will be disconnected.

ORDINARY GRAPHS

An *ordinary graph* (or simply *graph*) G is obtained by modifying a digraph D as follows. Both G and D have the same points and two points v_1 and v_2 in G are joined by an undirected line if and only if both the directed lines $v_1 \to v_2$ and $v_2 \to v_1$ occur in D. In order to avoid confusion between the notation for the lines of a digraph and the lines of a graph, we shall denote the lines of G by the variable symbols $y_1, y_2, \cdots$. We now develop a formula for the determinant of the variable adjacency matrix of G. Let $A(G)$ be the adjacency matrix of a graph G and let $A(G, y)$ be its variable adjacency matrix. We illustrate these matrices with the graph of Fig. 3.

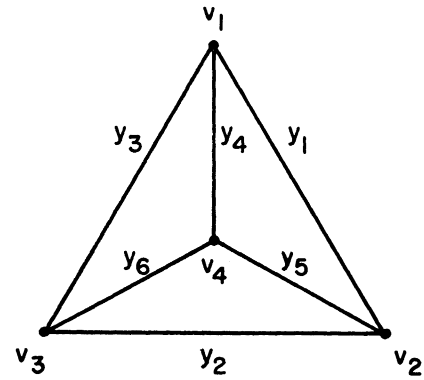

FIGURE 3

The variable adjacency matrix of this graph is

$$A(G, y) = \begin{bmatrix} 0 & y_1 & y_3 & y_4 \\ y_1 & 0 & y_2 & y_5 \\ y_3 & y_2 & 0 & y_6 \\ y_4 & y_5 & y_6 & 0 \end{bmatrix}.$$

We verify at once that the determinant of this matrix is

$$| A(G, y) | = y_1^2 y_6^2 + y_2^2 y_4^2 + y_3^2 y_5^2 - 2y_1y_2y_4y_6 - 2y_2y_3y_4y_5 - 2y_1y_3y_5y_6 ,$$

and

$$| A(G) | = -3.$$

It is clear that the variable adjacency matrix of an ordinary graph G may be obtained from that of the corresponding symmetric digraph D which has a symmetric pair of directed lines for each undirected line of G. We use this observation to define the properties of a subgraph of G whose lines correspond to a single nonvanishing term in the determinant $| A(G, y) |$.

A *path in a graph* G is the set of elements in an alternating sequence of points and (undirected) lines beginning and ending with a point, in which the points are distinct. A *cycle in a graph* is obtained from a path by adding the line joining the two endpoints of the path. A *spanning subgraph* of G has the same set of points as G. To illustrate, the set of all points of G, with no lines, is always a spanning subgraph of G. An *(ordinary) linear subgraph* of G is a spanning subgraph whose components are lines or cycles. Let n be the number of linear subgraphs of G and let G_i be the i'th linear subgraph. *An even component* of G_i has an even number of points. *L*et e_i be the number of even components of G_i and c_i be the number of components of G_i containing more than two points, and thus consisting of a single undirected cycle.

As in the case of digraphs, the variable determinant of an ordinary graph is the sum of the variable determinants of its ordinary linear subgraphs:

$$|A(G, y)| = \sum_{i=1}^{n} |A(G_i, y)|. \tag{5}$$

Therefore, it is sufficient to develop a formula for the variable determinant of an ordinary linear subgraph G_i. Let L_i be the set of components of G_i consisting of two points and the line joining them and let M_i be the remaining components of G_i each of which is a cycle.

THEOREM 3. *The variable determinant of an ordinary linear subgraph of an ordinary graph is given by the following formula.*

$$|A(G_i, y)| = (-1)^{e_i} 2^{c_i} \prod_{y_k \in L_i} y_k^2 \prod_{y_j \in M_i} y_j . \tag{6}$$

PROOF. This formula, although longer and more complicated in appearance than equation (1), is actually a corollary of that equation. We may see this by considering the digraph corresponding to the linear subgraph G_i obtained by replacing each undirected line y_k by the symmetric pair of directed lines joining the same two points, and calling each of these two directed lines by the same symbol y_k. The factor $(-1)^{e_i}$ of formula (6) is exactly the same as equation (1). It remains to show that the factor $p_i(x)$ becomes the rest of formula (6). To do this, it is necessary to point out that the linear subgraph G_i, after being converted to a symmetric digraph D_i, may contain several linear subgraphs. This is a consequence of the fact that every undirected cycle of G_i becomes the union of two directed cycles, only one of which is in any linear subgraph of D_i. Thus the number of linear subgraphs of the digraph just constructed from G_i is obtained by raising 2 to that power which is the number of components of G_i which are cycles, that is, contain more than two points. For each of the linear subgraphs of D_i, the product of the variables attached to the lines is given by multiplying all the variables attached to those lines of G_i which are contained in a cycle by the square of each variable attached to a line of G_i which is itself a component of G_i. This follows from the fact that each line of G_i in a cycle occurs just once in any ordinary linear subgraph while each variable attached to a line not in a cycle occurs twice in every such ordinary linear subgraph, once in each direction. Combining these observations, we obtain equation (6).

As immediate corollaries we obtain formulas for the determinant of the adjacency matrix of an ordinary linear subgraph:

$$|A(G_i)| = (-1)^{e_i} 2^{c_i}, \tag{7}$$

and also the determinant of the adjacency matrix of the original graph G:

$$|A(G)| = \sum_{i=1}^{n} (-1)^{e_i} 2^{c_i}. \tag{8}$$

In Figure 4 we show a graph G and its three ordinary linear subgraphs G_1 G_2, G_3. Since $|A(G_1, y)| = y_2^2 y_4^2$, $|A(G_2, y)| = y_1^2 y_3^2$, and

$$|A(G_3, y)| = -2y_1y_2y_3y_4,$$

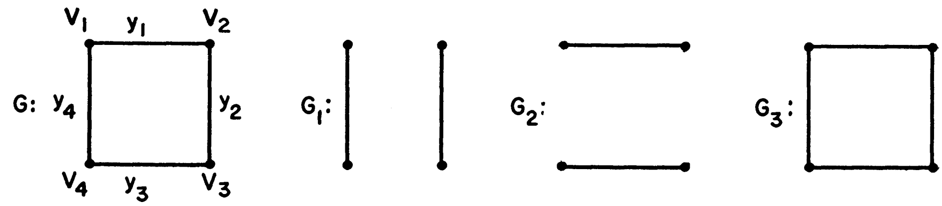

FIGURE 4

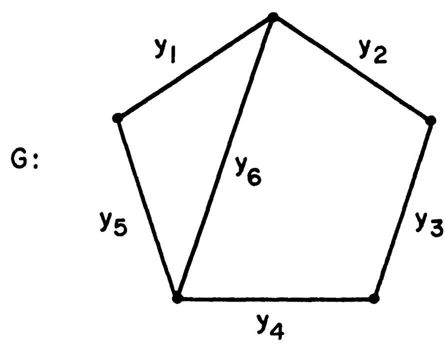

FIGURE 5

we see that

$$|A(G, y)| = y_1^2 y_3^2 + y_2^2 y_4^2 - 2y_1y_2y_3y_4$$

and

$$|A(G)| = 0.$$

Similarly, for the graph G of Fig. 5, we find

$$|A(G, y)| = 2y_1y_2y_3y_4y_5 - y_1y_5y_6y_3^2$$

and

$$|A(G)| = 1.$$

We conclude by observing that equation (1) in this article is of interest in electric network problems and has been anticipated in the literature. In the language of flow graphs used by Desoer [3] in his proof of Coates' formula [1], a "connection-gain" of a "flow graph" is the product of the line variables in a linear subgraph of a given digraph. In Fig. 5 of [3], a digraph with 4 points and its 5 linear subgraphs are depicted, and it is said: "Thus by simply listing all the connections, as is done on Fig. 5, one obtains all the terms of the sum" (where the sum is the value of a given determinant).

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