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# NOTE ON NONNEGATIVE MATRICES 

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Abstract. Let $A$ be a nonnegative square matrix and $B=$ $D_{1} A D_{2}$ where $D_{1}$ and $D_{2}$ are diagonal matrices with positive diagonal entries. Several proofs are known for the following theorem: If $A$ is fully indecomposable then $D_{1}$ and $D_{2}$ can be chosen so that $B$ is doubly stochastic. Moreover, $D_{1}$ and $D_{2}$ are unique up to a scalar factor. It is shown that these results can be easily obtained by considering a minimum of a certain rational function of several variables.

Several recent papers [1], [2], [3], [4] were devoted to the following problem: Given a nonnegative square matrix $A$, find the conditions for the existence of two diagonal matrices $D_{1}$ and $D_{2}$ such that $D_{1} A D_{2}$ is doubly stochastic. We shall show that it is related to a simple minimum problem. This leads to a short proof of Theorem (6.1) of [1] which avoids the use of Menon's operator.

We begin with some definitions. An $n \times n(n \geqq 2)$ matrix $A$ is reducible if there exists a permutation matrix $P$ such that

$$
P A P^{T}=\left(\begin{array}{ll}
A_{1} & 0 \\
B & A_{2}
\end{array}\right)
$$

where $A_{1}$ is a $k \times k$ matrix, $1 \leqq k \leqq n-1$. Otherwise we say that $A$ is irreducible.

An $n \times n(n \geqq 2)$ matrix $A$ is fully indecomposable if there do not exist permutation matrices $P$ and $Q$ such that

$$
P A Q=\left(\begin{array}{ll}
A_{1} & 0 \\
B & A_{2}
\end{array}\right)
$$

where $A_{1}$ is a $k \times k$ matrix, $1 \leqq k \leqq n-1$.
Theorem. Let $A$ be a nonnegative $n \times n$ fully indecomposable matrix. Then there exist diagonal matrices $D_{1}$ and $D_{2}$ with positive diagonals such that $D_{1} A D_{2}$ is doubly stochastic. Moreover $D_{1}$ and $D_{2}$ are uniquely determined up to scalar multiples.

For the proof we need the following
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Lemma. Let $A$ be a nonnegative $n \times n$ matrix. Then $A$ is fully indecomposable if and only if there exist permutation matrices $P$ and $Q$ such that $P A Q$ has a positive main diagonal and is irreducible.

A short proof of this lemma appears in [1].
Proof of the theorem. By the lemma we can assume that $A=\left(a_{i j}\right)$ has positive main diagonal and is irreducible. Let

$$
f\left(x_{1}, \cdots, x_{n}\right)=\prod_{k=1}^{n}\left(\sum_{i=1}^{n} a_{k i} x_{i}\right) / \prod_{k=1}^{n} x_{k}
$$

the variables being restricted by

$$
\begin{equation*}
x_{k}>0 \quad(1 \leqq k \leqq n), \quad \sum_{k=1}^{n} x_{k}=1 \tag{1}
\end{equation*}
$$

Let ( $b_{i}$ ) be a boundary point of the region (1) and, for instance, $b_{1}=\cdots=b_{s}=0, b_{k}>0(s<k \leqq n)$. Since $A$ is irreducible we infer that at least one entry $a_{i j}>0$ for $1 \leqq i \leqq s, s<j \leqq n$. This implies that $f\left(x_{1}, \cdots, x_{n}\right) \rightarrow+\infty$ when $\left(x_{k}\right) \rightarrow\left(b_{k}\right)$. Therefore $f$ attains its minimum in some point ( $c_{k}$ ) of the region (1). The partial derivatives of $f$ vanish at $\left(c_{k}\right)$ since $f$ is homogeneous. Hence,

$$
\sum_{k=1}^{n} c_{j} a_{k j}\left(\sum_{i=1}^{n} a_{k i} c_{i}\right)^{-1}=1 \quad(j=1, \cdots, n)
$$

which proves the first assertion of the theorem.
For the uniqueness it is sufficient to prove the following assertion: If the matrices $X=\left(x_{i j}\right), D_{1}=\operatorname{diag}\left(d_{1}^{\prime \prime}, \cdots, d_{n}^{\prime}\right), D_{2}=\operatorname{diag}\left(d_{1}^{\prime \prime}, \cdots, d_{n}^{\prime \prime}\right)$ satisfy
(i) $X$ is irreducible doubly stochastic with positive elements on the main diagonal;
(ii) $d_{i}^{\prime}>0, d_{i}^{\prime \prime}>0(1 \leqq i \leqq n)$;
(iii) $D_{1} X D_{2}$ is doubly stochastic, then $D_{1}$ and $D_{2}$ are scalar matrices.

Since

$$
\begin{aligned}
& \sum_{j=1}^{n} d_{i}^{\prime} x_{i j} d_{j}^{\prime \prime}=1 \Rightarrow\left(\max d_{i}^{\prime}\right)\left(\min d_{j}^{\prime \prime}\right) \leqq 1, \\
& \sum_{i=1}^{n} d_{i}^{\prime} x_{i j} d_{j}^{\prime \prime}=1 \Rightarrow\left(\max d_{i}^{\prime}\right)\left(\min d_{j}^{\prime \prime}\right) \leqq 1 .
\end{aligned}
$$

we conclude that none of these inequalities is strict. This implies that $x_{r s}=0$ whenever $d_{r}^{\prime}=\max d_{i}^{\prime}$ and $d_{s}^{\prime \prime}>\min d_{j}^{\prime \prime}$ or $d_{r}^{\prime}<\max d_{i}^{\prime}$ and $d_{s}^{\prime \prime}=\min d_{j}^{\prime \prime}$. This contradicts (i) unless $D_{1}$ and $D_{2}$ are scalar matrices.

The proof is completed.

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