

Note on Nonnegative Matrices

D. Ž. Djokovi#

Proceedings of the American Mathematical Society, Vol. 25, No. 1. (May, 1970), pp. 80-82.

Stable URL:

http://links.jstor.org/sici?sici=0002-9939%28197005%2925%3A1%3C80%3ANONM%3E2.0.CO%3B2-Q

Proceedings of the American Mathematical Society is currently published by American Mathematical Society.

Your use of the JSTOR archive indicates your acceptance of JSTOR's Terms and Conditions of Use, available at http://www.jstor.org/about/terms.html. JSTOR's Terms and Conditions of Use provides, in part, that unless you have obtained prior permission, you may not download an entire issue of a journal or multiple copies of articles, and you may use content in the JSTOR archive only for your personal, non-commercial use.

Please contact the publisher regarding any further use of this work. Publisher contact information may be obtained at <u>http://www.jstor.org/journals/ams.html</u>.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed page of such transmission.

The JSTOR Archive is a trusted digital repository providing for long-term preservation and access to leading academic journals and scholarly literature from around the world. The Archive is supported by libraries, scholarly societies, publishers, and foundations. It is an initiative of JSTOR, a not-for-profit organization with a mission to help the scholarly community take advantage of advances in technology. For more information regarding JSTOR, please contact support@jstor.org.

NOTE ON NONNEGATIVE MATRICES

D. Ž. DJOKOVIĆ¹

ABSTRACT. Let A be a nonnegative square matrix and $B = D_1AD_2$ where D_1 and D_2 are diagonal matrices with positive diagonal entries. Several proofs are known for the following theorem: If A is fully indecomposable then D_1 and D_2 can be chosen so that B is doubly stochastic. Moreover, D_1 and D_2 are unique up to a scalar factor. It is shown that these results can be easily obtained by considering a minimum of a certain rational function of several variables.

Several recent papers [1], [2], [3], [4] were devoted to the following problem: Given a nonnegative square matrix A, find the conditions for the existence of two diagonal matrices D_1 and D_2 such that D_1AD_2 is doubly stochastic. We shall show that it is related to a simple minimum problem. This leads to a short proof of Theorem (6.1) of [1] which avoids the use of Menon's operator.

We begin with some definitions. An $n \times n$ $(n \ge 2)$ matrix A is reducible if there exists a permutation matrix P such that

$$PAP^{T} = \begin{pmatrix} A_{1} & 0 \\ B & A_{2} \end{pmatrix}$$

where A_1 is a $k \times k$ matrix, $1 \le k \le n-1$. Otherwise we say that A is irreducible.

An $n \times n$ $(n \ge 2)$ matrix A is fully indecomposable if there do not exist permutation matrices P and Q such that

$$PAQ = \begin{pmatrix} A_1 & 0 \\ B & A_2 \end{pmatrix}$$

where A_1 is a $k \times k$ matrix, $1 \leq k \leq n-1$.

THEOREM. Let A be a nonnegative $n \times n$ fully indecomposable matrix. Then there exist diagonal matrices D_1 and D_2 with positive diagonals such that D_1AD_2 is doubly stochastic. Moreover D_1 and D_2 are uniquely determined up to scalar multiples.

For the proof we need the following

Received by the editors August 23, 1969.

AMS Subject Classifications. Primary 1560, 1565.

Key Words and Phrases. Nonnegative matrix, doubly stochastic matrix, irreducible matrix, fully indecomposable matrix.

¹ Supported in part by NRC Grant A-5285.

LEMMA. Let A be a nonnegative $n \times n$ matrix. Then A is fully indecomposable if and only if there exist permutation matrices P and Q such that PAQ has a positive main diagonal and is irreducible.

A short proof of this lemma appears in [1].

PROOF OF THE THEOREM. By the lemma we can assume that $A = (a_{ij})$ has positive main diagonal and is irreducible. Let

$$f(x_1, \cdots, x_n) = \prod_{k=1}^n \left(\sum_{i=1}^n a_{ki} x_i \right) / \prod_{k=1}^n x_k$$

the variables being restricted by

(1)
$$x_k > 0 \quad (1 \leq k \leq n), \qquad \sum_{k=1}^n x_k = 1.$$

Let (b_i) be a boundary point of the region (1) and, for instance, $b_1 = \cdots = b_s = 0, b_k > 0 \ (s < k \le n)$. Since A is irreducible we infer that at least one entry $a_{ij} > 0$ for $1 \le i \le s$, $s < j \le n$. This implies that $f(x_1, \cdots, x_n) \rightarrow +\infty$ when $(x_k) \rightarrow (b_k)$. Therefore f attains its minimum in some point (c_k) of the region (1). The partial derivatives of f vanish at (c_k) since f is homogeneous. Hence,

$$\sum_{k=1}^{n} c_{j} a_{kj} \left(\sum_{i=1}^{n} a_{ki} c_{i} \right)^{-1} = 1 \qquad (j = 1, \cdots, n)$$

which proves the first assertion of the theorem.

For the uniqueness it is sufficient to prove the following assertion: If the matrices $X = (x_{ij}), D_1 = \text{diag}(d'_1, \dots, d'_n), D_2 = \text{diag}(d''_1, \dots, d''_n)$ satisfy

(i) X is irreducible doubly stochastic with positive elements on the main diagonal;

(ii) $d'_i > 0, d''_i > 0 \ (1 \le i \le n);$

(iii) $D_1 X D_2$ is doubly stochastic, then D_1 and D_2 are scalar matrices. Since

$$\sum_{j=1}^{n} d'_i x_{ij} d''_j = 1 \Rightarrow (\max d'_i) (\min d''_j) \le 1,$$
$$\sum_{i=1}^{n} d'_i x_{ij} d''_j = 1 \Rightarrow (\max d'_i) (\min d''_j) \ge 1.$$

we conclude that none of these inequalities is strict. This implies that $x_{rs} = 0$ whenever $d'_r = \max d'_i$ and $d''_s > \min d''_j$ or $d'_r < \max d'_t$ and $d''_s = \min d''_j$. This contradicts (i) unless D_1 and D_2 are scalar matrices.

The proof is completed.

D. Ž. DJOKOVIĆ

References

1. R. A. Brualdi, S. V. Parter and H. Schneider, *The diagonal equivalence of a nonnegative matrix to a stochastic matrix*, J. Math. Anal. Appl. 16 (1966), 31-50. MR 34 #5844.

2. M. V. Menon, Reduction of a matrix with positive elements to a doubly stochastic matrix, Proc. Amer. Math. Soc. 18 (1967), 244-247. MR 35 #6708.

3. R. Sinkhorn, A relationship between arbitrary positive matrices and doubly stochastic matrices, Ann. Math. Statist. 35 (1964), 876–879. MR 28 #5072.

4. R. Sinkhorn and P. Knopp, Concerning nonnegative matrices and doubly stochastic matrices, Pacific J. Math. 21 (1967), 343-348. MR 35 #1617.

UNIVERSITY OF WATERLOO, ONTARIO, CANADA

http://www.jstor.org

LINKED CITATIONS

- Page 1 of 1 -



You have printed the following article:

Note on Nonnegative Matrices D. Ž. Djokovi# *Proceedings of the American Mathematical Society*, Vol. 25, No. 1. (May, 1970), pp. 80-82. Stable URL: http://links.jstor.org/sici?sici=0002-9939%28197005%2925%3A1%3C80%3ANONM%3E2.0.CO%3B2-Q

This article references the following linked citations. If you are trying to access articles from an off-campus location, you may be required to first logon via your library web site to access JSTOR. Please visit your library's website or contact a librarian to learn about options for remote access to JSTOR.

References

² Reduction of a Matrix with Positive Elements to a Doubly Stochastic Matrix
M. V. Menon
Proceedings of the American Mathematical Society, Vol. 18, No. 2. (Apr., 1967), pp. 244-247.
Stable URL:
http://links.jstor.org/sici?sici=0002-9939%28196704%2918%3A2%3C244%3AROAMWP%3E2.0.C0%3B2-X