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The Random-Walk Representation of Classical Spin Systems and Correlation Inequalities

II. The Skeleton Inequalities

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Abstract. We use the random-walk representation to prove the first few of a new family of correlation inequalities for ferromagnetic φ^4 lattice models. These inequalities state that the finite partial sums of the propagator-resummed perturbation expansion for the 4-point function form an alternating set of rigorous upper and lower bounds for the exact 4-point function. Generalizations to $2n$-point functions are also given. A simple construction of the continuum φ_d^4 quantum field theory ($d<4$), based on these inequalities, is described in a companion paper.

1. Introduction

This paper is a continuation of the work begun in preceding papers [1–3], where a random-walk expansion due originally to Symanzik [4, 5] (see also [6, 7]) is employed to derive a variety of correlation inequalities (among other results) for lattice models in classical statistical mechanics. The main result of [2] (see also [3] for a variant of the proof) is the new correlation inequality

$$0 \geqq u_4(x_1, x_2, x_3, x_4) \geqq -\sum_{z,z',z''} \langle \varphi_{x_1}\varphi_z\rangle \langle \varphi_{x_2}\varphi_z\rangle J_{zz'} J_{zz''} \langle \varphi_{z'}\varphi_{x_3}\rangle \langle \varphi_{z''}\varphi_{x_4}\rangle$$
$$- \text{two permutations } -\mathscr{E}, \tag{1.1}$$

where $\mathscr{E}$ is an extra term which turns out to be irrelevant in applications. This inequality implies [2, 3, 8–10] the *triviality* (i.e. Gaussianness) of the continuum limit for φ^4 or Ising models in dimension $d>4$. (For the Ising model this result was first obtained by Aizenman [8, 9], who proved a correlation inequality similar to (1.1) by graphical methods. A version of Aizenman's inequality also applies to the φ^4 model.)

In this paper we restrict attention to φ^4 models, and derive new correlation inequalities which will be (among other things) important ingredients in the proof of the *nontriviality* (i.e. non-Gaussianness) of the continuum limit for weakly coupled

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φ_d^4 field theories in dimension $d<4$. (This proof is carried out in a companion paper [11].) Of course, the existence and nontriviality of these superrenormalizable continuum field theories is already well known (see [12, 13] for $d=2$ and [14–24] for $d=3$); indeed, the construction for $d=3$ is one of the most difficult and subtle proofs ever devised in mathematical physics. Thus, the primary goal of the present work is to provide a simpler (though less powerful) construction of these models, roughly along the lines proposed previously by one of us [10]. For further discussion of this application, see the Introduction to [11].

What we prove in the present paper are in fact the first few of an entire family of correlation inequalities which can be described in words as follows: Consider the perturbation expansion (in powers of the bare coupling constant λ_0) for the connected 4-point function u_4 in the φ^4 lattice model. We now form the *propagator-resummed perturbation expansion* by considering only those graphs which contain no self-energy part, and by considering each line in these graphs to be the *exact* (interacting) 2-point function. Formally this is a partial resummation of ordinary perturbation theory. We then claim that the finite partial sums of this expansion form an alternating set of *rigorous upper and lower bounds* for the exact u_4, *valid for all* $\lambda_0 \geqq 0$. Actually, in this paper we give a complete proof only for the first three inequalities of this family (i.e. those to order 1, λ_0, and λ_0^2); although we are convinced that the entire family of inequalities is true, and we sketch briefly at the end of Sect. 3 how a proof ought to go, we must confess that the combinatorics required has been (up to now) beyond our ability.

Thus, the correlation inequalities proven in this paper are the following:

$$u_4(x_1, x_2, x_3, x_4) \leqq 0, \tag{1.2}$$

$$u_4(x_1, x_2, x_3, x_4) \geqq -\lambda_0 \sum_z \langle\varphi_{x_1}\varphi_z\rangle \langle\varphi_{x_2}\varphi_z\rangle \langle\varphi_{x_3}\varphi_z\rangle \langle\varphi_{x_4}\varphi_z\rangle, \tag{1.3}$$

$$\begin{aligned} u_4(x_1, x_2, x_3, x_4) \leqq & -\lambda_0 \sum_z \langle\varphi_{x_1}\varphi_z\rangle \langle\varphi_{x_2}\varphi_z\rangle \langle\varphi_{x_3}\varphi_z\rangle \langle\varphi_{x_4}\varphi_z\rangle \\ & + \frac{\lambda_0^2}{2} \sum_{z,z'} \langle\varphi_{x_1}\varphi_z\rangle \langle\varphi_{x_2}\varphi_z\rangle \langle\varphi_z\varphi_{z'}\rangle^2 \langle\varphi_{z'}\varphi_{x_3}\rangle \langle\varphi_{z'}\varphi_{x_4}\rangle \\ & + \text{two permutations}. \end{aligned} \tag{1.4}$$

Inequality (1.2) is, of course, the well-known Lebowitz inequality [25–29, 1, 3]. Inequalities (1.3) and (1.4) are improvements of inequalities proposed in [10, Eqs. (3.29) and (3.30)] and proven there subject to a conjectured correlation inequality on the 6-point function. We emphasize that no such conjecture is needed in the present work. An inequality of the type (1.3) is also a consequence of the work of Aizenman [9, Proposition 11.2].

Although (1.3) is similar in structure to (1.1) – both are "tree-graph bounds" – they have quite different meaning due to the different multiplying factors (λ_0 vs. J^2). Inequality (1.1) yields a *universal* (i.e. λ_0-independent) upper bound on the renormalized coupling constant g; this bound is excellent for $d>4$ (it implies triviality!), but is useless for $d<4$ (it is worse than the Glimm-Jaffe bound $g \leqq$ const [30], see also [31–33, 9]). Inequality (1.3), on the other hand, is not very useful for proving triviality in $d>4$, but is an excellent bound for superrenormalizable models in $d<4$ – as our analysis of these models [11] will show.

The plan of this paper is as follows: In Sect. 2, we recapitulate briefly the formalism of the random-walk expansion; more details can be found in [1] (see also [3] for a pedagogical introduction). The key new ingredient is a lemma on the "splitting of paths" (Lemma 2.1). In Sect. 3, we give a complete proof of inequalities (1.2)–(1.4) and a brief sketch of how one should be able to prove higher order inequalities. In Sect. 4, we derive some analogous inequalities for $2n$-point functions. In Sect. 5, we give a simple proof of a very strong form of the Gaussian inequality [34, 35], and derive as a corollary a truncated Gaussian inequality. Finally, in Sect. 6, we discuss briefly some extensions and applications.

2. Basic Formalism

In this section we briefly recapitulate the basic formalism of the random-walk expansion; see [1, 3] for more details. We consider a model of one-component classical spins on a finite lattice, with partition function

$$Z=\int e^{\frac{1}{2}(\varphi, J\varphi)}\prod_j g_j(\varphi_j^2)d\varphi_j. \tag{2.1}$$

Here J (called the "pair interaction") is a symmetric matrix, i.e. $J_{ij}=J_{ji}$. Beginning in Sect. 3, we shall require that $J_{ij}\geqq 0$ for all i,j ("ferromagnetism"); however, this assumption is unnecessary for the identities derived in the present section. We assume that each g_j is C^∞ and decays faster than exponentially at infinity along with all its derivatives. This very strong restriction on g_j (much stronger than really necessary) is made solely to avoid uninteresting technical problems; it can be removed by taking limits in the final formulae.

The 2-point function of our model is

$$\langle\varphi_x\varphi_y\rangle=Z^{-1}\int \varphi_x\varphi_y e^{\frac{1}{2}(\varphi, J\varphi)}\prod_j g_j(\varphi_j^2)d\varphi_j. \tag{2.2}$$

We insert into (2.2) the Fourier representation

$$g_j(\varphi_j^2)=\int e^{-ia_j\varphi_j^2}\hat{g}_j(a_j)da_j, \tag{2.3}$$

then interchange the order of integration and "half-perform" the now-Gaussian φ integral; the result is

$$\langle\varphi_x\varphi_y\rangle=Z^{-1}\int (2ia-J)_{xy}^{-1}e^{\frac{1}{2}(\varphi,(J-2ia)\varphi)}\prod_j d\varphi_j\hat{g}_j(a_j)da_j. \tag{2.4}$$

This interchange of integrals may appear somewhat dubious, but it yields a correct result because we can first move the contour of an integration in (2.3) to $\operatorname{Im} a_j=$ large negative constant, which makes the integrals absolutely convergent. Later we move the contour back again! This exploits the analyticity and decay of $\hat{g}_j(a_j)$, which is a consequence of our decay and smoothness assumptions on g_j.

Next we expand $(2ia-J)^{-1}$ in a Neumann series

$$(2ia-J)^{-1}=(2ia)^{-1}+(2ia)^{-1}J(2ia)^{-1}+\ldots \tag{2.5}$$

(which converges because of our distortion of a-integration contours). The sums over matrix indices implicit in (2.5) can be combined into one sum over a random walk; doing this and inserting into (2.4), we get

$$\langle\varphi_x\varphi_y\rangle = Z^{-1} \sum_{\omega:x\to y} J^\omega \int \left(\prod_j (2ia_j)^{-n_j(\omega)}\right) e^{\frac{1}{2}(\varphi,(J-2ia)\varphi)} \prod_j d\varphi_j \hat{g}_j(a_j) da_j . \quad (2.6)$$

Here the sum ranges over all walks $\omega = (\omega(0), \omega(1), \ldots, \omega(n))$ on the lattice starting at x and ending at y [i.e., $\omega(0) = x$, $\omega(n) = y$ with $n \geqq 0$ and $\omega(1), \ldots, \omega(n-1)$ arbitrary], $n_j(\omega)$ is the number of times that ω visits the site j, and

$$J^\omega = J_{\omega(0)\omega(1)} J_{\omega(1)\omega(2)} \ldots J_{\omega(n-1)\omega(n)} . \quad (2.7)$$

Using, for each site j, the identity

$$x^{-n} = \int_0^\infty \frac{e^{-tx} t^{n-1}}{(n-1)!} dt \quad (n \geqq 1) \quad (2.8)$$

in (2.6), we find

$$\langle\varphi_x\varphi_y\rangle = Z^{-1} \sum_{\omega:x\to y} J^\omega \int dv_\omega(t) e^{\frac{1}{2}(\varphi, J\varphi)} \prod_j d\varphi_j e^{-ia_j(\varphi_j^2 + 2t_j)} \hat{g}_j(a_j) da_j , \quad (2.9)$$

$$= Z^{-1} \sum_{\omega:x\to y} J^\omega \int dv_\omega(t) e^{\frac{1}{2}(\varphi, J\varphi)} \prod_j g_j(\varphi_j^2 + 2t_j) d\varphi_j , \quad (2.10)$$

where we have introduced the positive measure

$$dv_\omega(t) = \prod_j dv_{n_j(\omega)}(t_j) , \quad (2.11)$$

with

$$dv_n(s) = \begin{cases} \delta(s) ds & \text{if} \quad n = 0 , \\ \dfrac{s^{n-1}}{(n-1)!} \chi_{[0,\infty)}(s) ds & \text{if} \quad n \geqq 1 . \end{cases} \quad (2.12)$$

Now the φ-integral in (2.10) is precisely the partition function (2.1), except that each $g_j(\varphi_j^2)$ has been replaced by $g_j(\varphi_j^2 + 2t_j)$. Thus, defining

$$Z(t) = \int e^{\frac{1}{2}(\varphi, J\varphi)} \prod_j g_j(\varphi_j^2 + 2t_j) d\varphi_j , \quad (2.13)$$

and

$$\mathscr{Z}(t) = Z(t)/Z , \quad (2.14)$$

we have derived the fundamental formula

$$\langle\varphi_x\varphi_y\rangle = \sum_{\omega:x\to y} J^\omega \int dv_\omega(t) \mathscr{Z}(t) . \quad (2.15)$$

Similar formulas can be derived for $2n$-point functions; for example,

$$\langle\varphi_{x_1}\varphi_{x_2}\varphi_{x_3}\varphi_{x_4}\rangle = \sum_{\substack{\omega_1:x_1\to x_2 \\ \omega_2:x_3\to x_4}} J^{\omega_1+\omega_2} \int dv_{\omega_1}(t_1) dv_{\omega_2}(t_2) \mathscr{Z}(t_1+t_2)$$
$$+ \text{two permutations} \quad (2.16)$$
$$= \sum_{\omega:x_1\to x_2} J^\omega \int dv_\omega(t) \mathscr{Z}(t) \langle\varphi_{x_3}\varphi_{x_4}\rangle_t$$
$$+ \text{two permutations} , \quad (2.17)$$

where $\langle \cdot \rangle_t$ denotes normalized expectation with respect to the measure in (2.13), i.e.

$$\langle F(\varphi)\rangle_t = Z(t)^{-1} \int F(\varphi) e^{\frac{1}{2}(\varphi, J\varphi)} \prod_j g_j(\varphi_j^2 + 2t_j) d\varphi_j. \tag{2.18}$$

(We have also written $J^{\omega_1+\omega_2}$ as a convenient shorthand for $J^{\omega_1}J^{\omega_2}$.) In fact, (2.15), (2.17) and their generalizations to higher-point functions can be unified into the single integration-by-parts formula

$$\langle \varphi_x F(\varphi)\rangle = \sum_y \sum_{\omega : x \to y} J^\omega \int dv_\omega(t) \mathscr{Z}(t) \left\langle \frac{\partial F}{\partial \varphi_y} \right\rangle_t ; \tag{2.19}$$

see [1, 3] for the proof.

Finally, we present a lemma on the "splitting of paths" which we shall use repeatedly in what follows:

Lemma 2.1. *Let $j_1, \ldots, j_n$ be lattice sites, and let f be any (decent) function. Then*

$$\begin{aligned} &\sum_{\omega : x \to y} J^\omega \int dv_\omega(t) t_{j_1} \ldots t_{j_n} f(t) \\ &= \sum_{\pi \in \mathscr{P}_n} \sum_{\substack{\omega_0 : x \to j_{\pi(1)} \\ \omega_1 : j_{\pi(1)} \to j_{\pi(2)} \\ \vdots \\ \omega_n : j_{\pi(n)} \to y}} J^{\omega_0 + \ldots + \omega_n} \int dv_{\omega_0}(t_0) \ldots dv_{\omega_n}(t_n) f(t_0 + \ldots + t_n). \end{aligned} \tag{2.20}$$

Here $\mathscr{P}_n$ is the set of all permutations of $\{1, \ldots, n\}$.

Proof. We consider first the case $n = 1$. By (2.11) and (2.12), the measure $t_j dv_\omega(t)$ vanishes identically if the path ω never visits the site j; moreover if ω does visit j, then

$$t_j dv_\omega(t) = n_j(\omega) dv_{\omega'}(t), \tag{2.21}$$

where ω' is any path having

$$n_k(\omega') = \begin{cases} n_k(\omega) & \text{for} \quad k \neq j \\ n_k(\omega) + 1 & \text{for} \quad k = j. \end{cases} \tag{2.22}$$

(For example, ω' can be a walk obtained from ω by converting one of the visits to j into a double visit.) Now if ω_0 and ω_1 are any two paths such that

$$n_k(\omega') = n_k(\omega_0) + n_k(\omega_1) \text{ for all } k, \tag{2.23}$$

it follows easily from (2.11)/(2.12) [or from (2.8)] that

$$\int dv_{\omega'}(t) f(t) = \int dv_{\omega_0}(t_0) dv_{\omega_1}(t_1) f(t_0 + t_1). \tag{2.24}$$

So let ω_0 and ω_1 be the pieces of the path ω formed by splitting it at any one of its visits to the site j. These ω_0 and ω_1 satisfy (2.22)/(2.23): the extra visit to site j arises because j is now counted *both* as the final point of path ω_0 and as the initial point of path ω_1. Moreover, for each ω arising on the left side of (2.20) there are $n_j(\omega)$ ways of splitting it into ω_0 and ω_1 arising on the right side of (2.20). This exactly accounts for the factor $n_j(\omega)$ in (2.21), and completes the proof of the lemma for the case $n = 1$. (Note that $J^\omega = J^{\omega_0 + \omega_1}$ because there is no double-counting of bonds, only of sites.)

The general case now follows by induction. Indeed, assume that the lemma is true for $n=m$. We wish to show that it holds for $n=m+1$. Apply the case $n=m$ to the function

$$g(t)=t_{j_{m+1}}f(t). \tag{2.25}$$

Thus

$$\begin{aligned}
&\sum_{\omega:x\to y} J^{\omega}\int dv_{\omega}(t)t_{j_1}\dots t_{j_{m+1}}f(t)\\
&=\sum_{\omega:x\to y} J^{\omega}\int dv_{\omega}(t)t_{j_1}\dots t_{j_m}g(t)\\
&=\sum_{\pi\in\mathscr{P}_m}\sum_{\substack{\omega_0:x\to j_{\pi(1)}\\ \vdots\\ \omega_m:j_{\pi(m)}\to y}} J^{\omega_0+\dots+\omega_m}\int dv_{\omega_0}(t_0)\dots dv_{\omega_m}(t_m)g(t_0+\dots+t_m)\\
&=\sum_{\pi\in\mathscr{P}_m}\sum_{\substack{\omega_0:x\to j_{\pi(1)}\\ \vdots\\ \omega_m:j_{\pi(m)}\to y}} J^{\omega_0+\dots+\omega_m}\int dv_{\omega_0}(t_0)\dots dv_{\omega_m}(t_m)\\
&\quad\cdot[(t_0)_{j_{m+1}}+\dots+(t_m)_{j_{m+1}}]f(t_0+\dots+t_m).
\end{aligned}\tag{2.26}$$

Now apply the case $n=1$ to each of the functions $h_r(t_r)=f(t_0+\dots+t_m)$ $[0\leqq r\leqq m]$ with $\{t_s\}_{s\neq r}$ considered fixed; this splits the path ω_r at the site j_{m+1} and leaves all other paths $\{\omega_s\}_{s\neq r}$ unchanged. The sum of all these contributions is precisely the sum over $\pi'\in\mathscr{P}_{m+1}$ needed for the right side of (2.20). This completes the proof.

3. Bounds on u_4: Up to Second Order

We now assume that the pair interaction is ferromagnetic, i.e. $J_{ij}\geqq 0$ for all i,j. Moreover, we specialize to the case of a φ^4 model

$$g_j(\varphi^2)=\exp\left[-\frac{\lambda_0}{4!}\varphi^4-\frac{B_0}{2}\varphi^2\right] \tag{3.1}$$

($\lambda_0\geqq 0$) for all sites j. Then

$$g_j(\varphi^2+2t)=\exp\left[-\frac{\lambda_0}{4!}\varphi^4-\left(\frac{B_0}{2}+\frac{\lambda_0 t}{6}\right)\varphi^2-\left(\frac{\lambda_0 t^2}{6}+B_0 t\right)\right], \tag{3.2}$$

so that the primary effect of the t variables is to add a *space-dependent mass term* $\lambda_0 t_j\varphi_j^2/6$ to the Hamiltonian. (The t-dependent constant term $\lambda_0 t_j^2/6+B_0 t_j$ will affect the partition function $Z(t)$ but not the expectations $\langle\cdot\rangle_t$.) The crucial fact is that all variables t_j are nonnegative [by (2.12)], so that Griffiths' first and second inequalities [27] give

$$0\leqq\langle\varphi^A\rangle_t\leqq\langle\varphi^A\rangle_0 \tag{3.3}$$

for any product $\varphi^A=\prod_i \varphi_i^{A_i}$ of the spins φ_i. [Here $\langle\cdot\rangle_0$ is, of course, the same as $\langle\cdot\rangle$; we append the subscript 0 to emphasize that this is the expectation in the measure with all $t_j=0$. It should *not* be confused with a "free" (or Gaussian) expectation.]

Remark. It is allowable for λ_0 and B_0 to depend on the site j being considered. Indeed, a site-dependent B_0 is *essential* for the inductive proof given below (which uses lower-order inequalities applied to $\langle\cdot\rangle_t$ in place of $\langle\cdot\rangle_0$). A site-dependent λ_0 is optional; it would change the final results only by replacing λ_0 by $(\lambda_0)_z$ and λ_0^2 by $(\lambda_0)_z(\lambda_0)_{z'}$ in (1.3)/(1.4). We have pretended that λ_0 and B_0 are the same at all sites simply to lighten the notation.

We now consider the connected 4-point function (Ursell function)

$$\begin{aligned} u_4(x_1,x_2,x_3,x_4) \equiv {} & \langle\varphi_{x_1}\varphi_{x_2}\varphi_{x_3}\varphi_{x_4}\rangle - \langle\varphi_{x_1}\varphi_{x_2}\rangle\langle\varphi_{x_3}\varphi_{x_4}\rangle \\ & - \langle\varphi_{x_1}\varphi_{x_3}\rangle\langle\varphi_{x_2}\varphi_{x_4}\rangle - \langle\varphi_{x_1}\varphi_{x_4}\rangle\langle\varphi_{x_2}\varphi_{x_3}\rangle . \end{aligned} \tag{3.4}$$

By (2.17) and (2.15), this can be written as

$$u_4(x_1,x_2,x_3,x_4) = F(x_1,x_2|x_3,x_4) + F(x_1,x_3|x_2,x_4) + F(x_1,x_4|x_2,x_3), \tag{3.5}$$

with

$$F(x_1,x_2|x_3,x_4) = \sum_{\omega:x_1\to x_2} J^\omega \int dv_\omega(t)\mathscr{Z}(t)\left[\langle\varphi_{x_3}\varphi_{x_4}\rangle_t - \langle\varphi_{x_3}\varphi_{x_4}\rangle_0\right]. \tag{3.6}$$

Since J^ω, dv_ω, and $\mathscr{Z}(t)$ are all nonnegative, Griffiths' second inequality (3.3) implies that $F\leqq 0$, and hence $u_4\leqq 0$. This is the Lebowitz inequality (1.2), proved by the method of [1].

To get a lower bound on u_4, we examine more closely the bracket in (3.6). By the fundamental theorem of calculus,

$$\begin{aligned} \langle\varphi_{x_3}\varphi_{x_4}\rangle_t - \langle\varphi_{x_3}\varphi_{x_4}\rangle_0 &= \int_0^1 d\alpha \frac{d}{d\alpha}\langle\varphi_{x_3}\varphi_{x_4}\rangle_{\alpha t} \\ &= \int_0^1 d\alpha \sum_j \left(-\frac{\lambda_0}{6}\right) t_j \langle\varphi_{x_3}\varphi_{x_4};\varphi_j^2\rangle_{\alpha t}, \end{aligned} \tag{3.7}$$

where we have introduced the notation

$$\langle A;B\rangle \equiv \langle AB\rangle - \langle A\rangle\langle B\rangle . \tag{3.8}$$

Now

$$\begin{aligned} \langle\varphi_{x_3}\varphi_{x_4};\varphi_j^2\rangle_{\alpha t} &= 2\langle\varphi_{x_3}\varphi_j\rangle_{\alpha t}\langle\varphi_{x_4}\varphi_j\rangle_{\alpha t} + u_4(x_3,x_4,j,j)_{\alpha t} && (3.9) \\ &\leqq 2\langle\varphi_{x_3}\varphi_j\rangle_0\langle\varphi_{x_4}\varphi_j\rangle_0 && (3.10) \end{aligned}$$

by the Griffiths inequality (3.3) and the Lebowitz inequality $u_4\leqq 0$ (valid also for the theory $\langle\cdot\rangle_{\alpha t}$, since the αt is merely a mass term). The α integration is now trivial, and we conclude

$$\langle\varphi_{x_3}\varphi_{x_4}\rangle_t - \langle\varphi_{x_3}\varphi_{x_4}\rangle_0 \geqq -\frac{\lambda_0}{3}\sum_j t_j\langle\varphi_{x_3}\varphi_j\rangle_0\langle\varphi_{x_4}\varphi_j\rangle_0, \tag{3.11}$$

and hence

$$F(x_1,x_2|x_3,x_4) \geqq -\frac{\lambda_0}{3}\sum_j\left[\sum_{\omega:x_1\to x_2} J^\omega\int dv_\omega(t)\mathscr{Z}(t)t_j\right]\langle\varphi_{x_3}\varphi_j\rangle_0\langle\varphi_{x_4}\varphi_j\rangle_0 . \tag{3.12}$$

Finally, we use Lemma 2.1 to handle the t_j factor:

$$\begin{aligned}\sum_{\omega:x_1\to x_2} J^{\omega}\int dv_{\omega}(t)\mathscr{Z}(t)t_j &= \sum_{\substack{\omega_1:x_1\to j\\ \omega_2:j\to x_2}} J^{\omega_1+\omega_2}\int dv_{\omega_1}(t_1)dv_{\omega_2}(t_2)\mathscr{Z}(t_1+t_2)\\ &= \sum_{\omega_1:x_1\to j} J^{\omega_1}\int dv_{\omega_1}(t_1)\mathscr{Z}(t_1)\langle\varphi_j\varphi_{x_2}\rangle_{t_1}\\ &\leqq \sum_{\omega_1:x_1\to j} J^{\omega_1}\int dv_{\omega_1}(t_1)\mathscr{Z}(t_1)\langle\varphi_j\varphi_{x_2}\rangle_0\\ &= \langle\varphi_{x_1}\varphi_j\rangle_0\langle\varphi_j\varphi_{x_2}\rangle_0\,, \end{aligned}\tag{3.13}$$

where we have again used Griffiths' second inequality. Thus

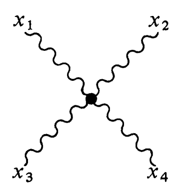
$$F(x_1,x_2|x_3,x_4)\geqq -\frac{\lambda_0}{3}\sum_j\langle\varphi_{x_1}\varphi_j\rangle_0\langle\varphi_{x_2}\varphi_j\rangle_0\langle\varphi_{x_3}\varphi_j\rangle_0\langle\varphi_{x_4}\varphi_j\rangle_0\,,\tag{3.14}$$

and

$$u_4(x_1,x_2,x_3,x_4)\geqq -\lambda_0\sum_j\langle\varphi_{x_1}\varphi_j\rangle_0\langle\varphi_{x_2}\varphi_j\rangle_0\langle\varphi_{x_3}\varphi_j\rangle_0\langle\varphi_{x_4}\varphi_j\rangle_0\,,\tag{3.15}$$

which is the "tree-graph lower bound" (1.3). In the Feynman-diagram notation [11, Sect. 3], (1.3)/(3.15) would be written

$$u_4(x_1,x_2,x_3,x_4)\geqq -\lambda_0 \tag{3.15'}$$



To prove the second-order bound (1.4), we analyze more carefully the terms thrown away by correlation inequalities in (3.10) and (3.13). Thus, we return to (3.9) and now seek a lower bound. For u_4 we use the tree-graph lower bound which we have just proved, namely

$$\begin{aligned}u_4(x_3,x_4,j,j)_{\alpha t} &\geqq -\lambda_0\sum_k\langle\varphi_{x_3}\varphi_k\rangle_{\alpha t}\langle\varphi_{x_4}\varphi_k\rangle_{\alpha t}\langle\varphi_j\varphi_k\rangle_{\alpha t}^2\\ &\geqq -\lambda_0\sum_k\langle\varphi_{x_3}\varphi_k\rangle_0\langle\varphi_{x_4}\varphi_k\rangle_0\langle\varphi_j\varphi_k\rangle_0^2\,.\end{aligned}\tag{3.16}$$

For each of the terms $\langle\varphi\varphi\rangle_{\alpha t}$ we use the lower bound (3.11); this generates four terms, of which we drop the last one (namely the one of order λ_0^2), which we are permitted to do since it is nonnegative. This gives

$$\begin{aligned}2\langle\varphi_{x_3}\varphi_j\rangle_{\alpha t}\langle\varphi_{x_4}\varphi_j\rangle_{\alpha t} \geqq\, & 2\langle\varphi_{x_3}\varphi_j\rangle_0\langle\varphi_{x_4}\varphi_j\rangle_0\\ &-\frac{2\lambda_0}{3}\langle\varphi_{x_3}\varphi_j\rangle_0\sum_k\alpha t_k\langle\varphi_{x_4}\varphi_k\rangle_0\langle\varphi_j\varphi_k\rangle_0\\ &-\frac{2\lambda_0}{3}\langle\varphi_{x_4}\varphi_j\rangle_0\sum_k\alpha t_k\langle\varphi_{x_3}\varphi_k\rangle_0\langle\varphi_j\varphi_k\rangle_0\,.\end{aligned}\tag{3.17}$$

We can now insert (3.16) and (3.17) into (3.9) and thence into (3.7), and perform the easy α integral; the result is

$$\begin{aligned}\langle\varphi_{x_3}\varphi_{x_4}\rangle_t-\langle\varphi_{x_3}\varphi_{x_4}\rangle_0 \leqq &-\frac{\lambda_0}{3}\sum_j t_j\langle\varphi_{x_3}\varphi_j\rangle_0\langle\varphi_{x_4}\varphi_j\rangle_0\\ &+\frac{\lambda_0^2}{6}\sum_{j,k} t_j\langle\varphi_{x_3}\varphi_k\rangle_0\langle\varphi_{x_4}\varphi_k\rangle_0\langle\varphi_j\varphi_k\rangle_0^2\\ &+\frac{\lambda_0^2}{18}\sum_{j,k} t_jt_k\langle\varphi_{x_3}\varphi_j\rangle_0\langle\varphi_{x_4}\varphi_k\rangle_0\langle\varphi_j\varphi_k\rangle_0\\ &+\frac{\lambda_0^2}{18}\sum_{j,k} t_jt_k\langle\varphi_{x_4}\varphi_j\rangle_0\langle\varphi_{x_3}\varphi_k\rangle_0\langle\varphi_j\varphi_k\rangle_0\,. \end{aligned}\tag{3.18}$$

(The last two terms in this formula can now be combined because they differ only by the labelling of dummy indices.) We now insert (3.18) into (3.6). The order-λ_0^2 terms in (3.18) are handled as before [see (3.13)]: we use Lemma 2.1 to split the path ω, then successively use (2.15) and Griffiths' second inequality to resum the random-walk expansion, bounding it from above by a product of 2-point functions. The result is

$$\begin{aligned}&\frac{\lambda_0^2}{6}\sum_{j,k}\langle\varphi_{x_1}\varphi_j\rangle_0\langle\varphi_{x_2}\varphi_j\rangle_0\langle\varphi_{x_3}\varphi_k\rangle_0\langle\varphi_{x_4}\varphi_k\rangle_0\langle\varphi_j\varphi_k\rangle_0^2\\ &\quad+\frac{\lambda_0^2}{9}\sum_{j,k}\langle\varphi_{x_1}\varphi_j\rangle_0\langle\varphi_{x_2}\varphi_k\rangle_0\langle\varphi_{x_3}\varphi_j\rangle_0\langle\varphi_{x_4}\varphi_k\rangle_0\langle\varphi_j\varphi_k\rangle_0^2\\ &\quad+\frac{\lambda_0^2}{9}\sum_{j,k}\langle\varphi_{x_1}\varphi_k\rangle_0\langle\varphi_{x_2}\varphi_j\rangle_0\langle\varphi_{x_3}\varphi_j\rangle_0\langle\varphi_{x_4}\varphi_k\rangle_0\langle\varphi_j\varphi_k\rangle_0^2\,.\end{aligned}\tag{3.19}$$

Finally, we consider the order-λ_0 term in (3.18); we must handle it a bit more carefully, because it is only of order λ_0, so there will be corrections to the tree graph of order λ_0^2 when the random-walk expansion is resummed [equivalently, the inequality (3.13) now goes in the wrong direction, since the term carries an overall minus sign]. We still use Lemma 2.1, but insert an inequality going in the opposite direction to (3.13):

$$\begin{aligned}\sum_{\omega:x_1\to x_2} J^\omega\int dv_\omega(t)\mathscr{Z}(t)t_j &=\sum_{\substack{\omega_1:x_1\to j\\ \omega_2:j\to x_2}} J^{\omega_1+\omega_2}\int dv_{\omega_1}(t_1)dv_{\omega_2}(t_2)\mathscr{Z}(t_1+t_2)\\ &=\sum_{\omega_1:x_1\to j} J^{\omega_1}\int dv_{\omega_1}(t_1)\mathscr{Z}(t_1)\langle\varphi_j\varphi_{x_2}\rangle_{t_1}\\ &=\langle\varphi_{x_1}\varphi_j\rangle_0\langle\varphi_j\varphi_{x_2}\rangle_0+F(x_1,j|j,x_2)\\ &\geqq\langle\varphi_{x_1}\varphi_j\rangle_0\langle\varphi_j\varphi_{x_2}\rangle_0\\ &\quad-\frac{\lambda_0}{3}\sum_k\langle\varphi_{x_1},\varphi_k\rangle_0\langle\varphi_{x_2}\varphi_k\rangle_0\langle\varphi_j\varphi_k\rangle_0^2\,,\end{aligned}\tag{3.20}$$

where we have used the definition (3.6) of F and the tree-graph lower bound (3.14).

Combining (3.18), (3.19), (3.20) and inserting them into (3.6), we conclude

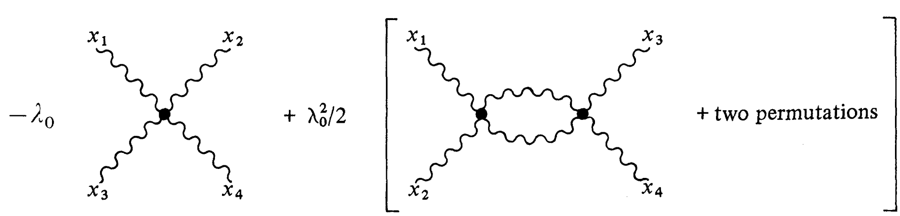
$$
\begin{aligned}
F(x_1,x_2|x_3,x_4) \leqq & -\frac{\lambda_0}{3}\sum_j \langle\varphi_{x_1}\varphi_j\rangle_0\langle\varphi_{x_2}\varphi_j\rangle_0\langle\varphi_{x_3}\varphi_j\rangle_0\langle\varphi_{x_4}\varphi_j\rangle_0 \\
& +\frac{5\lambda_0^2}{18}\sum_{j,k}\langle\varphi_{x_1}\varphi_j\rangle_0\langle\varphi_{x_2}\varphi_j\rangle_0\langle\varphi_j\varphi_k\rangle_0^2\langle\varphi_{x_3}\varphi_k\rangle_0\langle\varphi_{x_4}\varphi_k\rangle_0 \\
& +\frac{\lambda_0^2}{9}\sum_{j,k}\langle\varphi_{x_1}\varphi_j\rangle_0\langle\varphi_{x_3}\varphi_j\rangle_0\langle\varphi_j\varphi_k\rangle_0^2\langle\varphi_{x_2}\varphi_k\rangle_0\langle\varphi_{x_4}\varphi_k\rangle_0 \\
& +\frac{\lambda_0^2}{9}\sum_{j,k}\langle\varphi_{x_1}\varphi_j\rangle_0\langle\varphi_{x_4}\varphi_j\rangle_0\langle\varphi_j\varphi_k\rangle_0^2\langle\varphi_{x_2}\varphi_k\rangle_0\langle\varphi_{x_3}\varphi_k\rangle_0\,, \qquad (3.21)
\end{aligned}
$$

and hence

$$
\begin{aligned}
u_4(x_1,x_2,x_3,x_4) \leqq & -\lambda_0\sum_j \langle\varphi_{x_1}\varphi_j\rangle_0\langle\varphi_{x_2}\varphi_j\rangle_0\langle\varphi_{x_3}\varphi_j\rangle_0\langle\varphi_{x_4}\varphi_j\rangle_0 \\
& +\frac{\lambda_0^2}{2}\sum_{j,k}\langle\varphi_{x_1}\varphi_j\rangle_0\langle\varphi_{x_2}\varphi_j\rangle_0\langle\varphi_j\varphi_k\rangle_0^2\langle\varphi_{x_3}\varphi_k\rangle_0\langle\varphi_{x_4}\varphi_k\rangle_0 \\
& +\text{two permutations}. \qquad (3.22)
\end{aligned}
$$

This is precisely second-order propagator-resummed perturbation theory – even the coefficients are correct! In Feynman-diagram notation, (1.4)/(3.22) would be written

$u_4(x_1,x_2,x_3,x_4) \leqq$



It is now fairly clear how to extend this procedure to arbitrary order in λ_0. (It is also clear how tedious the computations will be!) The argument is inductive: at each order n, one produces both a bound on $\langle\varphi\varphi\rangle_t$ and a bound on F and u_4, each expressed in terms of $\langle\varphi\varphi\rangle_0$ (an upper bound for n even, a lower bound for n odd); for example, (3.3) and (3.6) ff. for $n=0$, (3.11) and (3.14)/(3.15) for $n=1$, (3.18) and (3.21)/(3.22) for $n=2$. Given all of the bounds of order $0\leqq k\leqq n-1$, one constructs the bounds of order n as follows:

1) In (3.9) one inserts the bound of order $n-1$ for $(u_4)_{\alpha t}$; the result is a sum of products of $\langle\varphi\varphi\rangle_{\alpha t}$, with coefficients of order λ_0^k with $0\leqq k\leqq n-1$.

2) In the term of order λ_0^k, one inserts everywhere the bound of order $n-1-k$ for $\langle\varphi\varphi\rangle_{\alpha t}$; these bounds will always have exactly the desired sign. The result is a sum of products of $\langle\varphi\varphi\rangle_0$ with also explicit factors of αt.

3) One now inserts this into (3.7) and performs the easy α integrations. The result is the bound of order n for $\langle\varphi\varphi\rangle_t$; it involves sums of products of $\langle\varphi\varphi\rangle_0$ with also explicit factors of $\lambda_0 t$. Unfortunately, terms of order higher than $(\lambda_0 t)^n$ do ap-

pear, if $n \geqq 2$. For $n=2$ this caused us no trouble, since the only such term was of order $(\lambda_0 t)^{n+1}$ and hence of a sign allowing it to be simply discarded [cf. (3.17) and (3.18)]. However, for $n \geqq 3$, higher-order correction terms of both signs will apparently occur and we do not know exactly what to do. It appears that one must keep these terms for the time being; later in the proof, one hopes, they may combine with other high-order terms and take on an unoffending sign.

4) One inserts the order-n bound for $\langle\varphi\varphi\rangle_t$ into (3.6) and uses Lemma 2.1 to handle the explicit factors of t. This produces yet more terms involving $\langle\varphi\varphi\rangle_{t'}$ and $F_{t'}$ which again have to be handled using the lower-order bounds on these quantities [cf. (3.19) and (3.20)]. And so on....

As the reader can see, the extension of our method to order $n \geqq 3$ is not entirely trivial. We have not pursued the matter, because for our main intended application – the construction of the φ_2^4 and φ_3^4 quantum field theories [11] – the inequalities of orders $n=0,1,2$ are sufficient. (In fact they suffice for φ_d^4 theories for any $d<10/3$; see Remark 1 at the end of Sect. 6 of [11].) We invite the reader to try to work out the case $n=3$. This should be a good warm-up toward constructing a proof to all orders.

Note added in proof. A. Bovier and G. Felder have recently proved the inequalities to all orders.

4. Bounds on 2*n*-Point Functions

In this section we derive analogues of the preceding inequalities for general $2n$-point functions. Let

$$H(x_1, x_2 | x_3, \ldots, x_{2n}) = \sum_{\omega: x_1 \to x_2} J^\omega \int dv_\omega(t) \mathscr{L}(t) \langle \varphi_{x_3} \cdots \varphi_{x_{2n}} \rangle_t . \tag{4.1}$$

[Note that here, unlike (3.6), we find it more convenient to consider *untruncated* correlation functions.] Then, by (2.19),

$$\langle \varphi_{x_1} \cdots \varphi_{x_{2n}} \rangle = \sum_{i=2}^{2n} H(x_1, x_i | x_2, \ldots, \not{x}_i, \ldots, x_{2n}), \tag{4.2}$$

where $\not{x}_i$ denotes that x_i has been deleted from the list. Since

$$\langle \varphi_{x_3} \cdots \varphi_{x_{2n}} \rangle_t \leqq \langle \varphi_{x_3} \cdots \varphi_{x_{2n}} \rangle_0 \tag{4.3}$$

by Griffiths' second inequality, we obtain immediately [using (2.15)] that

$$H(x_1, x_2 | x_3, \ldots, x_{2n}) \leqq \langle \varphi_{x_1} \varphi_{x_2} \rangle_0 \langle \varphi_{x_3} \cdots \varphi_{x_{2n}} \rangle_0 , \tag{4.4}$$

and hence

$$\langle \varphi_{x_1} \cdots \varphi_{x_{2n}} \rangle \leqq \sum_{i=2}^{2n} \langle \varphi_{x_1} \varphi_{x_i} \rangle_0 \langle \varphi_{x_2} \cdots \not{\varphi}_{x_i} \cdots \varphi_{x_{2n}} \rangle_0 . \tag{4.5}$$

This is the strong Gaussian inequality of Newman [34, 35, 9], proved by the method of [1]. By iterating (4.5) one can obtain the ordinary Gaussian inequality [34–36, 1],

$$\langle \varphi_{x_1} \cdots \varphi_{x_{2n}} \rangle \leqq \sum_{\text{pairings}} \prod \langle \varphi_{x_\alpha} \varphi_{x_\beta} \rangle . \tag{4.6}$$

However, in some applications the strong form (4.5) may be essential: compare, for example, the proof of the generalized Simon-Lieb-Rivasseau inequality as given in [1] with the inconclusive discussion in [37].

To obtain the first-order lower bound, we write

$$\begin{aligned}\langle\varphi_{x_3}\dots\varphi_{x_{2n}}\rangle_t &= \langle\varphi_{x_3}\dots\varphi_{x_{2n}}\rangle_0 + \int_0^1 d\alpha \frac{d}{d\alpha}\langle\varphi_{x_3}\dots\varphi_{x_{2n}}\rangle_{\alpha t} \\ &= \langle\varphi_{x_3}\dots\varphi_{x_{2n}}\rangle_0 + \int_0^1 d\alpha \sum_j \left(-\frac{\lambda_0}{6}\right) t_j \langle\varphi_{x_3}\dots\varphi_{x_{2n}};\varphi_j^2\rangle_{\alpha t}. \end{aligned} \tag{4.7}$$

Moreover, by the strong Gaussian inequality (4.5),

$$\begin{aligned}\langle\varphi_{x_3}\dots\varphi_{x_{2n}};\varphi_j^2\rangle_{\alpha t} &\leqq \sum_{i=3}^{2n} \langle\varphi_j\varphi_{x_i}\rangle_{\alpha t}\langle\varphi_j\varphi_{x_3}\dots\not\varphi_{x_i}\dots\varphi_{x_{2n}}\rangle_{\alpha t} \\ &\leqq \sum_{i=3}^{2n} \langle\varphi_j\varphi_{x_i}\rangle_0\langle\varphi_j\varphi_{x_3}\dots\not\varphi_{x_i}\dots\varphi_{x_{2n}}\rangle_0 . \end{aligned} \tag{4.8}$$

Inserting (4.7) and (4.8) into (4.1), and using the splitting lemma (Lemma 2.1) in the accustomed way, and then again using Griffiths' second inequality, we get

$$\begin{aligned} H(x_1,x_2|x_3,\dots,x_{2n}) &\geqq \langle\varphi_{x_1}\varphi_{x_2}\rangle_0\langle\varphi_{x_3}\dots\varphi_{x_{2n}}\rangle_0 \\ &- \frac{\lambda_0}{6}\sum_{i=3}^{2n} \langle\varphi_{x_1}\varphi_j\rangle_0\langle\varphi_{x_2}\varphi_j\rangle_0\langle\varphi_j\varphi_{x_i}\rangle_0\langle\varphi_j\varphi_{x_3}\dots\not\varphi_{x_i}\dots\varphi_{x_{2n}}\rangle_0 . \end{aligned} \tag{4.9}$$

Using (4.2) we find

$$\begin{aligned}\langle\varphi_{x_1}\dots\varphi_{x_{2n}}\rangle &\geqq \sum_{i=2}^{2n} \langle\varphi_{x_1}\varphi_{x_i}\rangle_0\langle\varphi_{x_2}\dots\not\varphi_{x_i}\dots\varphi_{x_{2n}}\rangle_0 \\ &- \frac{\lambda_0}{3}\sum_{\substack{i,k=2\\ i<k}}^{2n} \langle\varphi_{x_1}\varphi_j\rangle_0\langle\varphi_{x_i}\varphi_j\rangle_0\langle\varphi_{x_k}\varphi_j\rangle_0 \\ &\cdot\langle\varphi_j\varphi_{x_2}\dots\not\varphi_{x_i}\dots\not\varphi_{x_k}\dots\varphi_{x_{2n}}\rangle_0 . \end{aligned} \tag{4.10}$$

However, it is probably more convenient (although weaker) to apply the strong Gaussian inequality (4.5) to the last term in (4.9)/(4.10), yielding

$$\begin{aligned} H(x_1,x_2|x_3,\dots,x_{2n}) &\geqq \langle\varphi_{x_1}\varphi_{x_2}\rangle_0\langle\varphi_{x_3}\dots\varphi_{x_{2n}}\rangle_0 \\ &- \frac{\lambda_0}{3}\sum_{\substack{i,l=3\\ i<l}}^{2n} \langle\varphi_{x_1}\varphi_j\rangle_0\langle\varphi_{x_2}\varphi_j\rangle_0\langle\varphi_j\varphi_{x_i}\rangle_0\langle\varphi_j\varphi_{x_l}\rangle_0 \\ &\cdot\langle\varphi_{x_3}\dots\not\varphi_{x_i}\dots\not\varphi_{x_l}\dots\varphi_{x_{2n}}\rangle_0 , \end{aligned} \tag{4.11}$$

and

$$\begin{aligned}\langle\varphi_{x_1}\dots\varphi_{x_{2n}}\rangle &\geqq \sum_{i=2}^{2n} \langle\varphi_{x_1}\varphi_{x_i}\rangle_0\langle\varphi_{x_2}\dots\not\varphi_{x_i}\dots\varphi_{x_{2n}}\rangle_0 \\ &- \lambda_0\sum_j\sum_{\substack{i,k,l=2\\ i<k<l}}^{2n} \langle\varphi_{x_1}\varphi_j\rangle_0\langle\varphi_{x_i}\varphi_j\rangle_0\langle\varphi_{x_k}\varphi_j\rangle_0\langle\varphi_{x_l}\varphi_j\rangle_0 \\ &\cdot\langle\varphi_{x_2}\dots\not\varphi_{x_i}\dots\not\varphi_{x_k}\dots\not\varphi_{x_l}\dots\varphi_{x_{2n}}\rangle_0 . \end{aligned} \tag{4.12}$$

An alternative (though weaker still) form of these inequalities can be obtained by applying the Gaussian inequality (4.6) to the last term in (4.11)/(4.12), yielding

$$H(x_1, x_2|x_3, \ldots, x_{2n}) \geqq \langle \varphi_{x_1}\varphi_{x_2}\rangle_0 \langle \varphi_{x_3} \cdots \varphi_{x_{2n}}\rangle_0 - \frac{\lambda_0}{3} \sum_j \sum_{\substack{i,l=3 \\ i<l}}^{2n} \langle \varphi_{x_1}\varphi_j\rangle_0 \langle \varphi_{x_2}\varphi_j\rangle_0 \langle \varphi_j \varphi_{x_i}\rangle_0 \langle \varphi_j \varphi_{x_l}\rangle_0 \cdot \sum_{\substack{\text{pairings of} \\ \{x_3, \ldots, \not{x}_i, \ldots, \not{x}_l, \ldots, x_{2n}\}}} \prod \langle \varphi_{x_\alpha}\varphi_{x_\beta}\rangle_0 . \tag{4.13}$$

and

$$\langle \varphi_{x_1} \cdots \varphi_{x_{2n}}\rangle \geqq \sum_{i=2}^{2n} \langle \varphi_{x_1}\varphi_{x_i}\rangle_0 \langle \varphi_{x_2} \cdots \not{\varphi}_{x_i} \cdots \varphi_{x_{2n}}\rangle_0 - \lambda_0 \sum_{\substack{i,k,l=2 \\ i<k<l}}^{2n} \langle \varphi_{x_1}\varphi_j\rangle_0 \langle \varphi_{x_i}\varphi_j\rangle_0 \langle \varphi_{x_k}\varphi_j\rangle_0 \langle \varphi_{x_l}\varphi_j\rangle_0 \sum_{\substack{\text{pairings of} \\ \{x_2, \ldots, \not{x}_i, \ldots, \not{x}_k, \ldots, \not{x}_l, \ldots, x_{2n}\}}} \prod \langle \varphi_{x_\alpha}\varphi_{x_\beta}\rangle_0 . \tag{4.14}$$

(We beg the reader's indulgence if our attempts to find an understandable notation have not met with success.) Inequalities (4.9), (4.11), and (4.13) are the complementary bounds to (4.4); (4.10), (4.12), and (4.14) are the complementary bounds to (4.5). To derive a bound complementary to (4.6), we insert (4.14) [or (4.10) or (4.12)] repeatedly into itself, i.e., use (4.14) [or (4.10) or (4.12)] to get a lower bound on the $(2n-2)$-point function $\langle \varphi_{x_2} \cdots \not{\varphi}_{x_i} \cdots \varphi_{x_{2n}}\rangle_0$. The result is

$$\langle \varphi_{x_1} \cdots \varphi_{x_{2n}}\rangle \geqq \sum_{\substack{\text{pairings of} \\ \{x_1, \ldots, x_{2n}\}}} \prod \langle \varphi_{x_\alpha}\varphi_{x_\beta}\rangle - \lambda_0 \sum_j \sum_{\substack{i,k,l,m=1 \\ i<k<l<m}}^{2n} \langle \varphi_{x_i}\varphi_j\rangle \langle \varphi_{x_k}\varphi_j\rangle \langle \varphi_{x_l}\varphi_j\rangle \langle \varphi_{x_m}\varphi_j\rangle \cdot \sum_{\substack{\text{pairings of} \\ \{x_1, \ldots, \not{x}_i, \ldots, \not{x}_k, \ldots, \not{x}_l, \ldots, \not{x}_m, \ldots, x_{2n}\}}} \prod \langle \varphi_{x_\alpha}\varphi_{x_\beta}\rangle . \tag{4.15}$$

Inequality (4.15) is exactly first-order propagator-resummed perturbation theory: one picks four indices i, k, l, m out of the $2n$ points and connects them in a tree graph with the internal vertex j; the remaining $2n-4$ points are paired in all possible ways and connected using 2-point functions.

Remark. It would be interesting to know whether the tree graph with factor $-\lambda_0$ in (4.15) [or (4.11)–(4.14)] can be replaced by the actual $u_4(x_i, x_k, x_l, x_m)$; in view of (1.3) this would be an improvement. Such an inequality is proven by Aizenman [9, Proposition 12.1] with, however, a sub-optimal coefficient multiplying the u_4 ($\frac{2}{3}$ instead of 1). Even more interestingly, Aizenman proves a *reverse* bound of the same structure [with coefficient $2/n(n-1)$ multiplying the u_4]. Both these bounds are quite interesting because, unlike (4.15), they are *universal*, i.e., λ_0-independent. They give, for example, an explicit proof that for φ^4 or Ising models, $u_4 \equiv 0$ implies that the theory is Gaussian. This was first proven by Newman [38] using the Lee-Yang theorem. Aizenman's methods are, however, somewhat complicated (unlike his proof of (4.5) for the Ising model, which is exceedingly simple). It would be of interest, therefore, to study these same questions within the random-walk formalism. See also [2, 3].

Order-λ_0^2 bounds on the $2n$-point functions [analogous to (1.4)] can also be derived, but we shall leave these as an exercise for the reader.

5. More on the Gaussian Inequality

In [34] Newman proved, by graphical methods, a very general form of the Gaussian inequality: this general inequality includes (4.5) and (4.6) but has other interesting consequences as well. Subsequently, Sylvester [35] gave a slightly simpler proof, also using graphical methods. In this section we rederive Newman's result using the random-walk formalism. Actually, we prove a slight generalization, which Newman conjectured [34, Eq. (3.11)] but was unable to prove. Although our method of proof is quite different from Newman's, the underlying combinatoric structure is the same.

The first result, Proposition 5.1, is a corollary of the main theorem. We state it first, because it is easy to understand and because it will be used in our accompanying paper on the construction of φ_3^4 [11].

We consider models of the form (2.1); thus, expectations are given by

$$\langle F(\varphi)\rangle = Z^{-1} \int F(\varphi) e^{\frac{1}{2}(\varphi, J\varphi)} \prod_j g_j(\varphi_j^2) d\varphi_j , \tag{5.1}$$

where F is any (reasonable) function of the spins $\{\varphi_i\}$. We assume, as before, that $J_{ij} = J_{ji} \geqq 0$ for all i,j ("ferromagnetism"). Furthermore, we assume that each g_j is log concave (N.B.: *as a function of* φ_j^2, *not* φ_j) and decays faster than exponentially for φ_j^2 large. This includes, for example, the φ^4 model (3.1); more generally, it includes the Ellis-Monroe-Newman [28, 29] class

$$g_j(\varphi_j^2) = e^{-V_j(\varphi_j)} , \tag{5.2}$$

where each V_j is even and C^1 and grows faster than quadratically at infinity, with V_j' convex on $(0, \infty)$. The inclusion is strict, as can be shown by simple examples of φ^8 models. (A partially contrary statement made in [1] is incorrect.) Of course, limits of such models – for example, the spin-$\frac{1}{2}$ Ising model – can also be handled by taking limits in the final inequalities.

Remark. Since the graphical formalism of [34, 35, 9] applies only to the Ising model, the resulting proofs are valid only for models obtainable from the Ising model by the Griffiths-Simon "analog system" trick [39, 40, 9]. This class includes the φ^4 model but does not include the whole Ellis-Monroe-Newman class.

We now denote by $\langle \cdot \rangle^G$ the expectation corresponding to the *Gaussian* measure (of mean zero) whose covariance is the same as that of the system $\langle \cdot \rangle$. That is,

$$\langle \varphi_i \varphi_j \rangle^G = \langle \varphi_i \varphi_j \rangle , \tag{5.3a}$$

$$\langle \varphi_i \varphi_j \varphi_k \varphi_l \rangle^G = \langle \varphi_i \varphi_j \rangle \langle \varphi_k \varphi_l \rangle + \langle \varphi_i \varphi_k \rangle \langle \varphi_j \varphi_l \rangle + \langle \varphi_i \varphi_l \rangle \langle \varphi_j \varphi_k \rangle , \tag{5.3b}$$

and so on. Let $A=\{A_i\}$ be a multi-index, and let

$$\varphi^A=\prod_i \varphi_i^{A_i}. \tag{5.4}$$

Then the Gaussian inequality (4.6) states that

$$\langle\varphi^A\rangle \leqq \langle\varphi^A\rangle^G \tag{5.5}$$

for any multi-index A. Now it turns out that once-truncated expectations are also dominated by their Gaussian analogues. That is, let

$$\langle\varphi^A;\varphi^B\rangle \equiv \langle\varphi^A\varphi^B\rangle - \langle\varphi^A\rangle\langle\varphi^B\rangle, \tag{5.6}$$

and similarly for $\langle\cdot\rangle^G$. Then:

Proposition 5.1. *For any multi-indices A, B,*

$$\langle\varphi^A;\varphi^B\rangle \leqq \langle\varphi^A;\varphi^B\rangle^G. \tag{5.7}$$

Sketch of Proof. It is enough to prove Proposition 5.1 when A and B are even (otherwise the truncation is trivial, in which case the result is the ordinary Gaussian inequality). It is convenient to use the notation

$$\mathscr{Z}(\omega_1,\ldots,\omega_n) \equiv \left(\prod_{i=1}^n J^{\omega_i}\right)\int \prod_{i=1}^n d\nu_{\omega_i}(t_i)\mathscr{Z}(t_1+\ldots+t_n),$$

and to use explicit products, $\varphi_{x_1}\ldots\varphi_{x_m}$, instead of multi-indices. (Of course, the $x_1,\ldots,x_m$ need not all be distinct.) We set $X\equiv(x_1,\ldots,x_{2k})$, $Y\equiv(y_1,\ldots,y_{2l})$. As in Sect. 2, one may derive the identity

$$\begin{aligned}&\langle\varphi_{x_1}\ldots\varphi_{x_{2k}};\varphi_{y_1}\ldots\varphi_{y_{2l}}\rangle\\ &\quad=\sum_{\substack{\omega_1\ldots\omega_k:Y\leftarrow\\ \omega'_1\ldots\omega'_l:X\leftarrow}}[\mathscr{Z}(\omega_1,\ldots,\omega_k,\omega'_1,\ldots,\omega'_l)-\mathscr{Z}(\omega_1,\ldots,\omega_k)\mathscr{Z}(\omega'_1,\ldots,\omega'_l)]\\ &\quad\quad+\sum_{\substack{\omega_1\ldots\omega_\alpha:Y\leftarrow\\ \omega'_1\ldots\omega'_\beta:X\leftarrow\\ \omega''_1\ldots\omega''_\gamma:X\to Y}}\mathscr{Z}(\omega_1,\ldots,\omega_\alpha,\omega'_1,\ldots,\omega'_\beta,\omega''_1,\ldots,\omega''_\gamma),\end{aligned}$$

where $\alpha<k$, $\beta=l-(k-\alpha)$, $\gamma=2(k-\alpha)$. Here $\omega_1\ldots\omega_\alpha:X\leftarrow$ ranges over all possible choices of α walks whose endpoints are 2α distinct elements of X, and two such choices of α walks are considered to be identical if they differ merely by a re-ordering of the α walks and/or by interchanges of starting and ending points of one or more of those walks. The notation $\omega''_1\ldots\omega''_\gamma:X\to Y$ indicates that the starting point of ω''_j is in X and the ending point is in Y, for all $j=1,\ldots,\gamma$.

We note the inequality

$$\sum_{\omega_1\ldots\omega_k:W\leftarrow}\mathscr{Z}(\omega_1,\ldots,\omega_k,\omega'_1,\ldots,\omega'_l)\leqq\left\{\sum_{\omega_1\ldots\omega_k:W\leftarrow}\mathscr{Z}(\omega_1,\ldots,\omega_k)\right\}\mathscr{Z}(\omega'_1,\ldots,\omega'_l),$$

where W is a set of $2k$ points, $w_1,\ldots,w_{2k}$. This inequality can be derived by repeated application of Griffiths' second inequality, as in Sects. 3 and 4. By successive applications of this inequality we see that

$$\begin{aligned}\langle\varphi_{x_1}\ldots\varphi_{x_{2k}};\varphi_{y_1}\ldots\varphi_{y_{2l}}\rangle &\leqq \sum_{\substack{\omega_1\ldots\omega_\alpha:X\leftarrow\\ \omega'_1\ldots\omega'_\beta:Y\leftarrow\\ \omega''_1\ldots\omega''_\gamma:X\to Y}} \mathscr{Z}(\omega_1,\ldots,\omega_\alpha,\omega'_1,\ldots,\omega'_\beta,\omega''_1,\ldots,\omega''_\gamma)\\ &\leqq \sum_{\substack{\omega_1\ldots\omega_\alpha:X\leftarrow\\ \omega'_1\ldots\omega'_\beta:Y\leftarrow\\ \omega''_1\ldots\omega''_\gamma:X\to Y}} \mathscr{Z}(\omega_1)\ldots\mathscr{Z}(\omega_\alpha)\mathscr{Z}(\omega'_1)\ldots\mathscr{Z}(\omega'_\beta)\mathscr{Z}(\omega''_1)\ldots\mathscr{Z}(\omega''_\gamma)\\ &=\langle\varphi_{x_1}\ldots\varphi_{x_{2k}};\varphi_{y_1}\ldots\varphi_{y_{2l}}\rangle^G. \quad \square\end{aligned}$$

We now discuss the general Gaussian inequality (Theorem 5.2). It is convenient to continue to talk about explicit products $\varphi_{x_1}\ldots\varphi_{x_m}$ instead of multi-indices. Now fix an integer $n\geqq 1$, and let $\mathscr{C}$ be a class of partitions of the set $\{1,\ldots,2n\}$. Here $\mathscr{C}$ is said to be *admissible* (following [34]) if each way of partitioning $\{1,\ldots,2n\}$ into pairs is a refinement of some partition in $\mathscr{C}$. A trivial example of an admissible class is the class of all pair-partitions. As we shall see shortly, however, there exist many other interesting examples.

Theorem 5.2. *Let $\mathscr{C}$ be an admissible class of partitions of $\{1,\ldots,2n\}$. Then*

$$\langle\varphi_{x_1}\ldots\varphi_{x_{2n}}\rangle\leqq\sum_{\pi\in\mathscr{C}}\prod_{Y\in\pi}\Big\langle\prod_{i\in Y}\varphi_i\Big\rangle. \tag{5.8}$$

Proof. We first introduce some notation (which is by far the worst part of this subject): Let $\mathscr{P}$ be the set of all partitions of $\{1,\ldots,2n\}$ into pairs. Now let

$$\pi_0=\{\{i_1,j_1\},\{i_2,j_2\},\ldots,\{i_n,j_n\}\}$$

be one of those pair partitions. Let $\boldsymbol{\omega}=(\omega_1,\ldots,\omega_n)$ be a family of walks on the lattice. We then say that $\boldsymbol{\omega}$ is *subordinate to* π_0, which we write as $\boldsymbol{\omega}\prec\pi_0$, in case

$$\begin{gathered}\omega_1:x_{i_1}\to x_{j_1}\\ \omega_2:x_{i_2}\to x_{j_2}\\ \vdots\\ \omega_n:x_{i_n}\to x_{j_n}.\end{gathered}$$

(Note that since reorderings of the n pairs, or the switching of i and j within one or more pairs, do not create a distinct partition, the same is considered to be true of the $\boldsymbol{\omega}$; that is, $\boldsymbol{\omega}$ should be considered to be an *unordered* family of *unoriented* walks. Alternatively, we could consider $\boldsymbol{\omega}$ to be an ordered family of oriented walks and then divide everywhere by the factor $2^n n!$.)

Using the random-walk formalism (see Sect. 2), it is now easy to derive the identity

$$\langle\varphi^X\rangle=\sum_{\pi_0\in\mathscr{P}}\sum_{\boldsymbol{\omega}\prec\pi_0}J^{\boldsymbol{\omega}}\int d\nu_{\boldsymbol{\omega}}(\mathbf{t})\mathscr{Z}(\hat{t}). \tag{5.9}$$

Here φ^X is shorthand for $\prod_{i=1}^{2n}\varphi_{x_i}$, $J^{\boldsymbol{\omega}}$ for $\prod_{i=1}^{n}J^{\omega_i}$, $d\nu_{\boldsymbol{\omega}}(\mathbf{t})$ for $\prod_{i=1}^{n}d\nu_{\omega_i}(t_i)$, and $\hat{t}$ for $\sum_{i=1}^{n}t_i$. For example, the case $n=2$ of (5.9) is just (2.16). Now let $\mathscr{C}$ be an admissible class of partitions of $\{1,\ldots,2n\}$. Since every $\pi_0\in\mathscr{P}$ refines at least one $\pi\in\mathscr{C}$ (we denote this by $\pi_0 \prec \pi$), it follows from (5.9) (and the positivity of everything in sight) that

$$\langle \varphi^X \rangle \leqq \sum_{\pi \in \mathscr{C}} \sum_{\substack{\pi_0 \in \mathscr{P} \\ \pi_0 \prec \pi}} \sum_{\omega \prec \pi_0} J^{\omega} \int d\nu_{\omega}(\mathbf{t}) \mathscr{Z}(\hat{t}). \tag{5.10}$$

Now fix a partition $\pi = \{I_1, \ldots, I_m\}$. Since $\boldsymbol{\omega}$ is subordinate to π_0 which refines π, we can decompose $\boldsymbol{\omega}$ into subfamilies $\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_m$ (each consisting of one or more walks) corresponding to the sets $I_1, \ldots, I_m$. Similarly we decompose $\mathbf{t}$ into subfamilies $\mathbf{t}_1, \ldots, \mathbf{t}_m$ with corresponding partial sums $\hat{t}_1, \ldots, \hat{t}_m$; clearly $\hat{t} = \hat{t}_1 + \ldots + \hat{t}_m$. Note now that

$$J^{\omega} = J^{\omega_1} \ldots J^{\omega_m}, \tag{5.11}$$

$$d\nu_{\omega}(\mathbf{t}) = d\nu_{\omega_1}(\mathbf{t}_1) \ldots d\nu_{\omega_m}(\mathbf{t}_m), \tag{5.12}$$

and

$$\mathscr{Z}(\hat{t}) = \frac{Z(\hat{t}_1 + \ldots + \hat{t}_m)}{Z(\hat{t}_2 + \ldots + \hat{t}_m)} \frac{Z(\hat{t}_2 + \ldots + \hat{t}_m)}{Z(\hat{t}_3 + \ldots + \hat{t}_m)} \ldots \frac{Z(\hat{t}_m)}{Z} \tag{5.13}$$

[recall (2.14)]. We now perform the sum over $\boldsymbol{\omega}$ in (5.10) in steps, starting first with the subfamily $\boldsymbol{\omega}_m$, then $\boldsymbol{\omega}_{m-1}$, etc., through $\boldsymbol{\omega}_1$. At each stage, we use the by now familiar relation

$$\sum_{\imath} J^{\omega_i} \int d\nu_{\omega_i}(\mathbf{t}_i) \frac{Z(\hat{t}_i + \ldots + \hat{t}_m)}{Z(\hat{t}_{i+1} + \ldots + \hat{t}_m)} = \langle \varphi^{X_i} \rangle_{\hat{t}_{i+1} + \ldots + \hat{t}_m} \leqq \langle \varphi^{X_i} \rangle, \tag{5.14}$$

where φ^{X_i} is shorthand for $\prod_{j \in I_i} \varphi_{x_j}$, the equality is a resummation based on a multi-spin analogue of (2.15), and the inequality is Griffiths' second inequality (which applies here because of the hypothesis on the single-spin distribution g_j; see [1]). Collecting results, we get precisely (5.8). □

We sympathize with the reader who is by now totally mesmerized by the notational and combinatoric complexity – this proof was painful for us to write out, too. But let us emphasize again that the underlying ideas are extremely simple: they are nothing more than the ideas involved in our proof of the Lebowitz inequality (3.4)–(3.6) ff. or the Gaussian inequality (4.4)–(4.6) ff.

Example 1. Let $\{I_1, I_2\}$ be a partition of $\{1, \ldots, 2n\}$ into two subsets. The class of partitions consisting of this single partition is not admissible, because any partition of $\{1, \ldots, 2n\}$ into pairs where one or more pairs "join" I_1 and I_2 (i.e., one element of a pair is in I_1 and the other in I_2) cannot be a refinement of $\{I_1, I_2\}$. However, we can produce an admissible class $\mathscr{C}$ by supplementing $\{I_1, I_2\}$ with all partitions of the form

$$\{I_1 \backslash Z_1, I_2 \backslash Z_2, P_1, \ldots, P_l\}, \tag{5.15}$$

where $Z_1 \subset I_1$, $Z_2 \subset I_2$, $|Z_1| = |Z_2| = l \geqq 1$, and $P_1, \ldots, P_l$ are pairs each of which has one element taken from Z_1 and the other taken from Z_2. Using this admissible class in Theorem 5.2, we get the following strong form of the truncated Gaussian inequality:

Corollary 5.3.

$$\langle \varphi^{X_1};\varphi^{X_2}\rangle \leqq \sum_{Z_1\subset X_1}\sum_{Z_2\subset X_2}\sum_{\gamma:Z_1\to Z_2}\langle \varphi^{X_1\setminus Z_1}\varphi^{X_2\setminus Z_2}\rangle \prod_{i\in Z_1}\langle \varphi_{x_i}\varphi_{x_{\gamma(i)}}\rangle\,, \tag{5.16}$$

where $|Z_1|=|Z_2|\geqq 1$ *and* γ *is summed over all bijections from* Z_1 *to* Z_2.

Example 2. As in the preceding example, let $\{I_1, I_2\}$ be a partition of $\{1, \ldots, 2n)$ into two subsets. We now produce an admissible class $\mathscr{C}'$ by supplementing $\{I_1, I_2\}$ with all partitions of the form

$$\{P_1, \ldots, P_n\}\,, \tag{5.17}$$

where $P_1, \ldots, P_n$ are pairs, at least one of which has one element taken from I_1 and the other from I_2. Using this admissible class in Theorem 5.2, we recover the ordinary truncated Gaussian inequality, Proposition 5.1. (Alternatively, Proposition 5.1 can be derived by applying the ordinary Gaussian inequality (4.6) to $\langle \varphi^{X_1\setminus Z_1}\varphi^{X_2\setminus Z_2}\rangle$ in Corollary 5.3.)

Finally, we sketch the proof of a first-order skeleton inequality for

$$\langle \varphi_{x_1}\ldots\varphi_{x_{2k}};\varphi_{y_1}\ldots\varphi_{y_{2l}}\rangle\,,$$

complementary to Proposition 5.1. Let H denote an arbitrary Feynman diagram with a single internal vertex of order 4 and with external vertices at the elements of $X\cup Y$, which connects at least one element of X to at least one element of Y. Let I_H denote the Feynman amplitude corresponding to H, with propagators given by the exact two-point function, $\langle \varphi_x\varphi_y\rangle$.

Proposition 5.4.

$$\langle \varphi_{x_1}\ldots\varphi_{x_{2k}};\varphi_{y_1}\ldots\varphi_{y_{2l}}\rangle \geqq \langle \varphi_{x_1}\ldots\varphi_{x_{2k}};\varphi_{y_1}\ldots\varphi_{y_{2l}}\rangle^G - \lambda_0\sum_H I_H\,.$$

The proof is a straightforward, but notationally cumbersome, combination of the arguments leading to (4.14) and those of the proof of Proposition 5.1.

6. Assorted Remarks

(1) *Two-component models.* The results established in this paper can be extended to two-component $\lambda_0|\boldsymbol{\varphi}|^4$-models:

Firstly, all the results proven in previous sections are valid for two-component isotropic ferromagnets (with some changes in combinatoric coefficients in the second-order skeleton inequalities) if φ is replaced by φ^1, the 1-component of $\boldsymbol{\varphi}$. The proofs are virtually identical to the one-component case, once one knows that

$$\langle \varphi^1_{x_1}\ldots\varphi^1_{x_n}\rangle_t \leqq \langle \varphi^1_{x_1}\ldots\varphi^1_{x_n}\rangle_0 \tag{6.1}$$

whenever $t_j \geqq 0$, for all j. This in turn is an immediate consequence of the Ginibre inequality [41, 42],

$$\langle \varphi^1_{x_1}\ldots\varphi^1_{x_n};|\boldsymbol{\varphi}_j|^2\rangle_t \geqq 0\,. \tag{6.2}$$

More generally, one can establish numerous correlation inequalities for mixed expectations $\langle \varphi^1_{x_1} \dots \varphi^1_{x_k} \varphi^2_{y_1} \dots \varphi^2_{y_l} \rangle$ and certain truncated versions thereof. For example, using the random-walk formulation, it is extremely easy to show that

$$\langle \varphi^1_{x_1} \varphi^1_{x_2} ; \varphi^2_{y_1} \varphi^2_{y_2} \rangle \leqq 0 , \tag{6.3}$$

an inequality first proven in [43]. The methods of proving all these inequalities are essentially identical to those used in Sects. 3–5, for the one-component case; for example, (6.3) is proven by the arguments leading to (3.6) ff. The key fact is that

$$\begin{aligned} \sum_{\omega_1 \dots \omega_k} \left(\prod_{i=1}^{k} J^{\omega_i} \right) \int \prod_{i=1}^{k} dv_{\omega_i}(t_i) \mathscr{Z}(t_1 + \dots + t_k + s) \\ \leqq \prod_{i=1}^{k} \left[\sum_{\omega_i} J^{\omega_i} \int dv_{\omega_i}(t_i) \mathscr{Z}(t_i) \right] \cdot \mathscr{Z}(s) ; \end{aligned} \tag{6.4}$$

this is a consequence of the Ginibre inequality (6.2). [There are more general versions of (6.4), proven by the same arguments, which we refrain from stating.] The beauty of the random-walk formalism, in the isotropic case, is that the quantities $\mathscr{Z}(t)$ make no reference to internal indices; the only effect of internal indices is to restrict the class of pairings (endpoints of random walks) entering into the sum over random walks; only like indices can be paired.

One can also develop a random-walk representation for models with anisotropic pair interaction, $J^1_{ij} \varphi^1_i \varphi^1_j + J^2_{ij} \varphi^2_i \varphi^2_j$, $J^1_{ij} \geqq |J^2_{ij}|$, and/or anisotropic φ^4 coupling [e.g., $\lambda_0 (\varphi^1_x)^2 (\varphi^2_x)^2$]. In the latter case, the variables t_j carry internal indices. One can prove numerous correlation inequalities, some of which go in the reverse direction from the usual isotropic case.

Finally, the correlation inequalities discussed in this paper *would* extend to general N-component $\lambda_0 |\boldsymbol{\varphi}|^4$ models *if* the Ginibre inequality were known for these models. Unfortunately it is known, at present, only for $N = 1, 2$ [44].

(2) *Edwards model (self-suppressing walk).* The Edwards model [45] of self-suppressing walks is a simplified description of the excluded-volume effects in polymer physics. Correlation functions in the Edwards model are defined as follows:

$$G_{2n}(x_1, y_1, \dots, x_n, y_n) = \sum_{\substack{\omega_i : x_i \to y_i \\ i = 1, \dots, n}} \mathscr{Z}(\omega_1, \dots, \omega_n) , \tag{6.5}$$

where

$$\mathscr{Z}(\omega_1, \dots, \omega_n) = \left(\prod_{i=1}^{n} J^{\omega_i} \right) \int \prod_{i=1}^{n} dv_{\omega_i}(t_i) \mathscr{Z}(t_1 + \dots + t_n) , \tag{6.6}$$

and

$$\mathscr{Z}(t) = \prod_j e^{-(\lambda_0/6) t_j^2 - B_0 t_j} ; \qquad \lambda_0 > 0 . \tag{6.7}$$

We note that

$$\sum_p G_{2n}(x_{p(1)}, x_{p(2)}, \dots, x_{p(2n-1)}, x_{p(2n)}) ,$$

where p ranges over all pairings, is analogous to the correlation function $\langle\varphi_{x_1}\dots\varphi_{x_{2n}}\rangle$ in a one-component $\lambda_0\varphi^4$-model; however, in the Edwards model, $\mathscr{Z}(t)$ is given *explicitly* by (6.7). The key identity in the analysis of the Edwards model is

$$\mathscr{Z}(t+s)=\mathscr{Z}(t)\mathscr{Z}(s)\exp\left[-2\lambda_0\sum_j t_js_j\right], \tag{6.8}$$

from which follow

$$\mathscr{Z}(t+s)\leqq\mathscr{Z}(t)\mathscr{Z}(s) \tag{6.9}$$

and

$$\mathscr{Z}(t+s)\geqq\mathscr{Z}(t)\mathscr{Z}(s)\left[1-2\lambda_0\sum_j t_js_j\right]. \tag{6.10}$$

From (6.9) we obtain the analogue of the Lebowitz and the Gaussian inequalities, and from (6.10) we deduce first-order skeleton inequalities. Higher-order skeleton inequalities have been established recently by A. Bovier, G. Felder et al. (private communication).

(3) *Skeleton vs. universal bounds.* Let $\mathscr{Z}(t)$ be as in (2.14) ($\lambda_0\varphi^4$-model), or as in (6.7) (Edwards model). We define

$$\mathring{\mathscr{Z}}(t)=\prod_j e^{(\lambda_0/6)t_j^2+B_0t_j}\mathscr{Z}(t). \tag{6.11}$$

We claim that if $t_j\geqq0$, $s_j\geqq0$ for all j,

$$\mathring{\mathscr{Z}}(t+s)\geqq\mathring{\mathscr{Z}}(t)\mathring{\mathscr{Z}}(s). \tag{6.12}$$

In the Edwards model this is an equality. In the $\lambda_0\varphi^4$-model,

$$\ln\mathring{\mathscr{Z}}(t+s)=\ln\mathring{\mathscr{Z}}(t)+\int_0^1\frac{\frac{\partial}{\partial\alpha}\mathring{\mathscr{Z}}(t+\alpha s)}{\mathring{\mathscr{Z}}(t+\alpha s)}.$$

Clearly, by (2.13)/(2.14) and (3.2),

$$\frac{\frac{\partial}{\partial\alpha}\mathring{\mathscr{Z}}(t+\alpha s)}{\mathring{\mathscr{Z}}(t+\alpha s)}=-\frac{\lambda_0}{6}\sum_j s_j\langle\varphi_j^2\rangle_{t+\alpha s},$$

and, by the second Griffiths inequality,

$$-\frac{\lambda_0}{6}\sum_j s_j\langle\varphi_j^2\rangle_{t+\alpha s}\geqq-\frac{\lambda_0}{6}\sum_j s_j\langle\varphi_j^2\rangle_{\alpha s}=\frac{\frac{\partial}{\partial\alpha}\mathring{\mathscr{Z}}(\alpha s)}{\mathring{\mathscr{Z}}(\alpha s)},$$

for positive λ_0. Thus

$$\begin{aligned}\ln\mathring{\mathscr{Z}}(t+s)&\geqq\ln\mathring{\mathscr{Z}}(t)+\int_0^1\frac{\frac{\partial}{\partial\alpha}\mathring{\mathscr{Z}}(\alpha s)}{\mathring{\mathscr{Z}}(\alpha s)}d\alpha\\&=\ln\mathring{\mathscr{Z}}(t)+\ln\mathring{\mathscr{Z}}(s),\end{aligned} \tag{6.13}$$

proving (6.12). By (6.11),

$$\mathscr{Z}(t+s) \geqq \mathscr{Z}(t)\mathscr{Z}(s) \prod_j e^{-(\lambda_0/3)t_j s_j}; \tag{6.14}$$

see [2]. Now, note that by (2.16) and (3.4),

$$\begin{aligned} u_4(x_1,x_2,x_3,x_4) &= \sum_p \sum_{\substack{\omega_1: x_{p(1)} \to x_{p(2)} \\ \omega_2: x_{p(3)} \to x_{p(4)}}} J^{\omega_1} J^{\omega_2} \int dv_{\omega_1}(t) dv_{\omega_2}(s) \, [\mathscr{Z}(t+s) - \mathscr{Z}(t)\mathscr{Z}(s)] \\ &\geqq \sum_p \sum_{\substack{\omega_1: x_{p(1)} \to x_{p(2)} \\ \omega_2: x_{p(3)} \to x_{p(4)}}} J^{\omega_1} J^{\omega_2} \int dv_{\omega_1}(t) dv_{\omega_2}(s) \\ &\quad \cdot \mathscr{Z}(t)\mathscr{Z}(s) \left[\prod_j e^{-(\lambda_0/3)t_j s_j} - 1\right]. \end{aligned} \tag{6.15}$$

If we now use the bound

$$\prod_j e^{-(\lambda_0/3)t_j s_j} - 1 \geqq -\frac{\lambda_0}{3} \sum_j t_j s_j, \tag{6.16}$$

and apply the splitting lemma, we obtain the first-order skeleton inequality, (1.3). The universal lower bound, (1.1), follows by inserting the bound

$$\prod_j e^{-(\lambda_0/3)t_j s_j} - 1 \geqq \begin{cases} -1 & \text{if} \quad \omega_1 \cap \omega_2 \neq \emptyset \\ 0 & \text{otherwise}, \end{cases} \tag{6.17}$$

and using a Simon-Lieb type inequality; see [2, 3]. Aizenman [9, Proposition 11.2] also has a bound which unifies the universal and first-order skeleton inequalities. Similar arguments work also for $2n$-point functions.

(4) *Correction of an error in* [1]. A. Holtkamp and E. B. Dynkin (private communications) have independently pointed out to us that the equation asserted in Lemma 1.2 of [1] is incorrect. The correct formula is

$$\det(\Lambda - J)^{-1} = (\det \Lambda)^{-1} \exp\left[\sum_j \sum_{\omega: j \to j} \frac{J_\omega}{|\omega|} \prod_j \lambda_i^{-n(i,\omega)}\right]. \tag{6.18}$$

The difference in the formulas arises from walks ω which traverse some loop several times. This inaccuracy does not affect any of the theorems of [1].

(5) Dynkin [46] has reformulated the ideas of [1] in a more probabilistic language.

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